

MATH

(4)

- Probability & statistics } → Eng. maths
 - Combinatorics
 - Graph theory
 - Set theory & algebra
 - Logic
- } Discrete maths
- Linear algebra
 - Num. methods
 - Calculus.
- } Engineering maths

Recommended Book - (Kenneth Rosen)

1. Lectures : Theory & WB
2. WK Book
3. PS - I & II
4. Rosen : DM
5. EM for GATE - By Sundaram Sir.

Probability & Statistics

- Basic probab.
- Probab. Distribution
- Statistics.

Basic probab. -

- Science of uncertain events.

1. Experiment
- Random - Can't predict before.
 - Non-random - can be predicted before result comes out.

2. Sample space: Set of all possible outcomes.

$$S = \{H, T\}, \quad S = \{1, 2, 3, 4, 5, 6\}$$

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

$$S = \{(1, 1), (1, 2), \dots, (1, 6), \\ (2, 1), (2, 2), \dots, \\ \dots, \dots, \dots, (6, 6)\}$$

3. Event

Event \subseteq Sample space

- event is any subset of sample space.
- Sample space is universal set.

Eg. $S = \{H, T\}$

$$E_1 = \{ \}$$

$$E_2 = \{H\}$$

$$E_3 = \{T\}$$

$$E_4 = \{H, T\}$$

$$\Rightarrow \boxed{p(E) = \frac{n(E)}{n(S)}} \rightarrow \text{Classical probability eq}^n$$

Only when all outcomes are equally-likely i.e. all possibility has same chance to come. In dice, (1, - - - 6) no., all have equally-likely property.

- In purely ~~class~~ lottery system, we can use

It.

$$p(E_1) = \frac{0}{2} = 0 \rightarrow \text{impossible event}$$

$$p(E_2) = \frac{1}{2} = p(E_3)$$

$$p(E_4) = \text{either head or tail}$$

$$= \frac{2}{2} = 1 \rightarrow \text{Sure event.}$$

Eg. in case of dice - $2^6 = 64$ events are possible.

When dice $\rightarrow 2^{36}$ events will happen.

prob. of max is 3 is \rightarrow

$$p(\text{max}=3) = \frac{5}{36}$$

(1, 2, 3, 3)

(3, 1, 2, 3)

(3, 3)

\downarrow
should not be counted twice.

Probability Approaches -

- Classical
- Frequency

\downarrow How to calculate

- Classical - $p(E) = \frac{n(E)}{n(S)}$

- Frequency - $p(E) = \frac{f(E)}{\sum f} = \frac{f(E)}{n}$

Classical assumption -

- Sample space is finite.
- Outcomes must be equally-likely.

⇒ Classical - Analytical / Theoretical → Exact
Frequency - Practical → Approximate

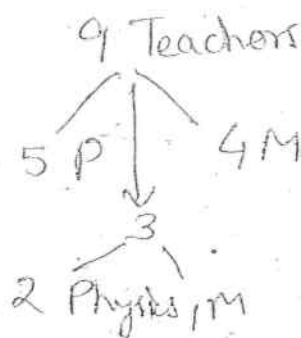
	f	p = $\frac{f}{n}$
A	10	.10 → 10% chance of A grade.
B	20	.20
C	60	.60
D	10	.10
	100	

- In word "BIRD", how many will start by D.

$\boxed{D} \text{ III} \rightarrow 3! = 6$

- $p(\text{start by D}) = \frac{n(\text{start by D})}{n(S)}$

↳ permutation = $\frac{3!}{4!} = \frac{1}{4}$



4 Maths ⇒ $5C_2 \times 4C_1$

→ Combinatory problem.

Now

p(

5. Pro

1.

2.

3.

Maths

In ca

also

This

6. Types

- eq

- mu

- co

- in

Now prob. of 2P & 1M \Rightarrow

$$P(2P+1M) = \frac{{}^5C_2 \times {}^4C_1}{{}^9C_3} \rightarrow \left. \begin{array}{l} \text{Conditional} \\ \text{Unconditional} \end{array} \right\} \text{Is known as } \downarrow \text{ probability basically.}$$

5. Probability Axioms -

1. $0 \leq p \leq 1$
2. $p(S) = 1$
3. $p(A \cup B) = p(A) + p(B) \rightarrow$ Only when A & B are mutually exclusive

Mutually exclusive - Can't happen together.

$$\text{i.e., } (A \cap B) = \emptyset$$

$$\text{So, } p(A \cap B) = 0.$$

$$\text{Hence, } p(A \cup B) = p(A) + p(B).$$

In case of cards - kings & Hearts. They are not mutually exclusive, because hearts also have king.

This case is called - joint probability.

6. Types of events -

- equally likely
- mutually exclusive
- collectively exhaustive
- independent.

Eg. $A = \{1, 2, 3\}$ \rightarrow equally likely
 $B = \{4, 5, 6\}$

Mutually exclusive -

1. $A \cap B = \phi$

2. $P(A \cap B) = 0$

3. $P(A \cup B) = P(A) + P(B)$

$A = \{1, 2, 3\}$ \rightarrow eq^y like.
 $B = \{3, 4, 6\}$ \rightarrow not mutually exclusive

Collectively exhaustive -

1. $A \cup B = S$

2. $P(A \cup B) = 1$

Mutually exclusive & collectively exhaustive -

\Rightarrow $P(A) + P(B) = 1$

$\Rightarrow P(B) = 1 - P(A)$

Q. A & B are running race. $P(A) = 0.1$ then what is $P(B) = ?$

If "only" A & B \rightarrow collectively exhaustive.

then $P(B) = 1 - P(A)$
 $= 1 - 0.1 = 0.9$

Eg. dice day

Again

Take table of dice ^{as} previously -

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + \dots + 6 \times \frac{1}{6}$$

$$= \frac{21}{6} = 3.5 \text{ due to symmetry}$$

Now,

$$V(X) = \sum x^2 p(x) - (\sum x \cdot p(x))^2$$

$$= (40^2 \times 0.1 + 50^2 \times 0.5 + 60^2 \times 0.1 + 70^2 \times 0.3) - (56)^2$$

$$V(X) = \dots$$

Bienayme - Chernyshev Rule -

$$p(\mu - k\sigma \leq X \leq \mu + k\sigma) \geq 1 - \frac{1}{k^2} \quad ; k > 1$$

like

$$p(56 - 20 \leq X \leq 56 + 20)$$

$$p(36 \leq X \leq 76) \geq 1 - \frac{1}{2^2}$$

$$= \frac{3}{4} \rightarrow 75\% \text{ chance at least}$$

prob. of $k\sigma$ is always $\geq 1 - \frac{1}{k^2}$

\Rightarrow 75% is chance of not losing money in market.
also known as what is risk?

- mi
rule
 \sum

Prob. Or
expe
and

\Rightarrow

p_i

$E(X)$

\Rightarrow
an exam

$=$

$$2. \quad p(x=5) = \frac{3}{15} = \frac{1}{5} \quad / \quad \frac{1}{6}$$

$$3. \quad p(x \geq 5) = \frac{2}{6} = \frac{1}{3}$$

$$4. \quad p(x \leq 5) = \frac{5}{6}$$

$$5. \quad p(4 \leq x \leq 6) = \frac{3}{6} = \frac{1}{2}$$

} in case
of
dice.

6. $E(x) \rightarrow$ Expected value of x .

$$E(x) = \mu_x = \bar{x} = \text{Avg. value of } x$$

$$\Rightarrow E(x) = \sum x \cdot p(x)$$

7. $V(x) \rightarrow$ Variable values. \rightarrow Variance.

$$V(x) = \sum x^2 p(x) - \left(\sum x p(x) \right)^2$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2$$

$$V(x) = \sigma_x^2 \Rightarrow \sigma_x = \sqrt{V(x)}$$

Standard deviation

$$E(g(x)) = \sum g(x) \cdot p(x)$$

$$E(x^2) = \sum x^2 \cdot p(x)$$

$$E(x^3) = \sum x^3 \cdot p(x)$$

$$\sum (x^{n+1}) = \sum (x^{n+1}) \cdot p(x) \dots$$

Independent -

1. $P(A|B) = P(A)$ \longrightarrow Given that B already happened
 \searrow conditional probab.

Unconditional probab. - marginal probab.

i.e. cond. prob is same as uncond. prob.

2. $P(B|A) = P(B)$

3. $P(A \cap B) = P(A) \cdot P(B)$

i.e. if B is happening, it does not effect prob. of A.

As, we know,

$$P(A \cap B) = P(A) * P(B|A)$$

In general \longrightarrow

$$P(A \cap B) = P(B) * P(A|B)$$

\therefore cond. prob = uncond. prob, then,

$$P(B) = P\left(\frac{B}{A}\right)$$

Eg.

	A	B
dice	6	<u>6</u>
day	I	<u>II</u>

 \Rightarrow independent case.

$$\begin{aligned} \Rightarrow P\left(\frac{6}{I} \cap \frac{6}{II}\right) &= P\left(\frac{6}{I}\right) * P\left(\frac{6}{II}\right) \\ &= \frac{1}{6} * \frac{1}{6} = \frac{1}{36} \end{aligned}$$

Again, as, if

	A	A
	6	6
	I	II

 \rightarrow Not independent case.

$$\text{Then } \Rightarrow P\left(\frac{A}{I}\right) * P\left(\frac{A}{II} / \frac{A}{I}\right)$$

$$= \frac{4}{52} \times \frac{3}{51}$$

Eg. When take 2 cards with replacement then, becomes independent.

$$\Rightarrow P\left(\frac{A}{I}\right) * P\left(\frac{A}{II} / \frac{A}{I}\right)$$

$$= \frac{4}{52} \times \frac{4}{52}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) \times P(B|A)$$

$$\Rightarrow P(A \cap B) \leq P(A)$$

$$\Rightarrow P(A \cup B) \leq P(A) + P(B)$$

$$\Rightarrow P(A \cap B) \leq P(B|A)$$

$$P(A|B) \geq P(A \cap B)$$

Rules of probability -

$$1. P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$2. P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$$

$$3. P(A^c) = 1 - P(A) \rightarrow \text{Complementary event}$$

$$\Rightarrow A \cup A^c = S$$

$$P(A^c \cap B^c) = 1 - P(A \cup B)$$

$$P(A^c \cup B^c) = 1 - P(A \cap B)$$

Q. $P(A) = 0.1$, $P(B) = 0.2$, $P(A \cap B) = 0.05$

What is neither condition case?

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [0.1 + 0.2 - 0.05] \\ &= 0.75 \text{ neither of them} \\ &\quad \text{prob.} \end{aligned}$$

Q. 2 dice thrown either of them is not 6.

$$P\left(\frac{6}{I} \cup \frac{6}{II}\right) = ?$$

let us assume 1st for, $P\left(\frac{6}{I}\right)$.

$$\begin{aligned} \Rightarrow P\left(\frac{6^c}{I} \cup \frac{6^c}{II}\right) &= P\left(\left(\frac{6}{I} \cap \frac{6}{II}\right)^c\right) \\ &= 1 - P\left(\frac{6}{I} \cap \frac{6}{II}\right) \\ &= 1 - \left(\frac{1}{6} \times \frac{1}{6}\right) \\ &= 1 - \frac{1}{36} = \frac{35}{36} \end{aligned}$$

is prob. that neither ~~is~~
comes with 6.

$$A^c \wedge B^c \rightarrow \text{NOR}$$

$$A^c \cup B^c \rightarrow \text{NAND}$$

Q. Let us take all possible words from, "MISSISSIPPI" ?

$$p(\text{start with } s) = 1 - p(\text{start without } s)$$

$$= 1 - \frac{n(\text{start with } s)}{n(s)}$$

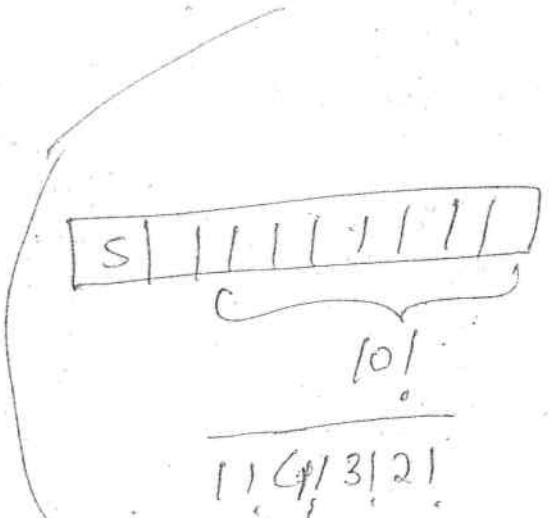
- M-1
- I-4
- S-4
- P-2

- 11

$$\Rightarrow \frac{11!}{1!4!4!2!} = n(s)$$

$$\frac{10!}{4!3!2!}$$

$$\Rightarrow = 1 - \frac{11!}{2!4!4!}$$



- M-1
- I-4
- S-3
- P-2

- 10

$$\frac{11!}{4!3!2!}$$

$$\Rightarrow = 1 - \frac{4}{11} = \frac{7}{11}$$

Q.

Solⁿ

5. day

$$4. P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow \text{when event already happened}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Q. $P(\text{rain}) = 0.1$

$P(\text{humid} \& \text{rain}) = 0.05$

What is $P(\text{humid}|\text{rain}) = ?$

Solⁿ

$$P(\text{humid}|\text{rain}) = \frac{P(H \cap R)}{P(R)}$$

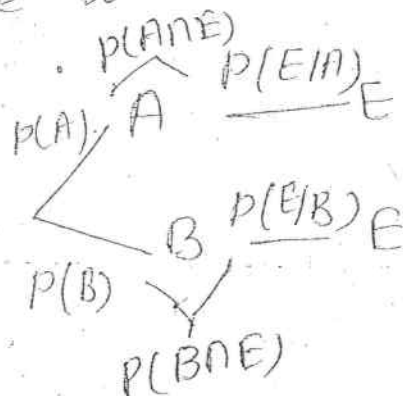
humid on day which is raining

$$= \frac{0.05}{0.1} = \frac{1}{2}$$

$P(\text{humid} \& \text{Rain}) = P(\text{humid}) \times P(\text{Rain}|\text{humid})$

↓
since $P(\text{humid})$ is not given, so, we can't use this further more.

5. Where both can happen at some particular day.



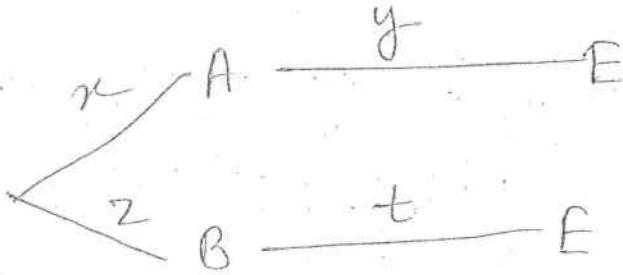
Rules of total probability -

$$P(E) = P(E \cap A) + P(E \cap B)$$

as, E can happen with A as well as B.

$$= P(A) \times P(E/A) + P(B) \times P(E/B)$$

In general \rightarrow



$$\Rightarrow \boxed{P(E) = xy + zt}$$

6. Bayes' Theorem - Given that E is already happened. Then have to calculate $P(A)$ or $P(B)$ in this case. i.e. given $P(A/E)$ & $P(B/E)$.

$$\text{As, } P(A/E) = \frac{P(A \cap E)}{P(E)}$$

$P(E)$ can be obtain by "Rule of total prob".

$$\Rightarrow P(A/E) = \frac{xy}{xy + zt}$$

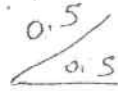
Similarly,

$$P(B|E) = \frac{P(B \cap E)}{P(E)} = \frac{z + t}{xy + zt}$$

↑ brand
↓
Total prob.

always \downarrow

$$P(A|E) + P(B|E) + P(C|E) = 1$$



Problems - (on Rule no. 5 & 6)

Prob.

Bag 1

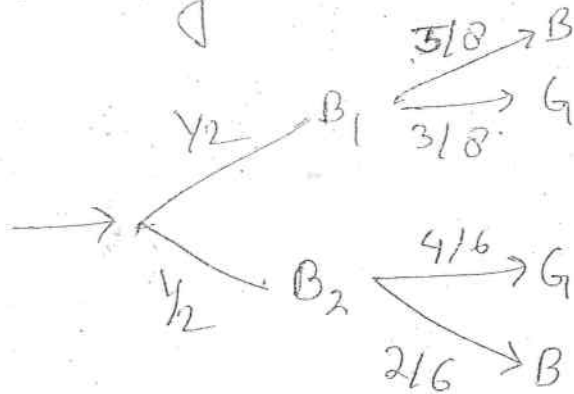
Bag 2

5B, 3G

2B, 4G

pick one bag at random, what is $P(G)$?

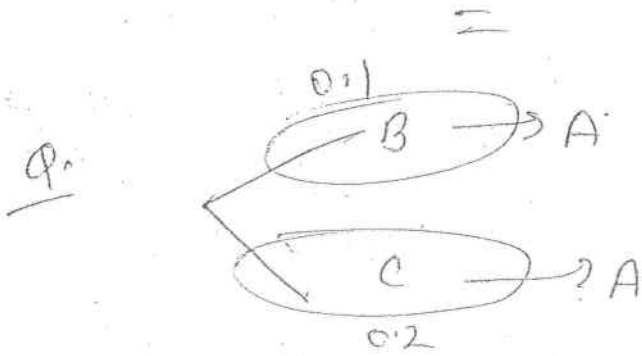
Solⁿ



$$\Rightarrow P(G) = \frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{4}{6}$$
$$= \frac{1}{2} \left(\frac{3}{8} + \frac{4}{6} \right)$$

Q. he founds, ball is green, what is chance it is from bag 1.

$$\Rightarrow P\left(\frac{B_1}{G}\right) = \frac{P(B_1 \cap G)}{P(G)} = \frac{\frac{1}{2} \times \frac{3}{8}}{\frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{4}{6}}$$



$$P(A \cap B) = 0.1, \quad P(A \cap C) = 0.2$$

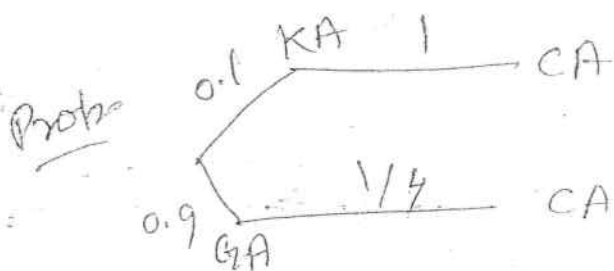
what is, $P(B/A) = ?$

Solⁿ

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{0.1}{P(A)}$$

$$\therefore P(A) = 0.1 + 0.2 = 0.3$$

$$P(B/A) = \frac{0.1}{0.3} = \frac{1}{3}$$



0.1 Knows Ans.

0.9 Guess Ans.

what is prob. he knows correct ans?
as 'if ques has 4 options.

⇒ Find out $P(KA|CA) = ?$

$$\Rightarrow P(KA|CA) = \frac{P(KA \cap CA)}{P(CA)}$$

$$= \frac{0.1 \times 1}{0.1 \times 1 + 0.9 \times \frac{1}{4}} = \frac{0.1}{0.1 + 0.225}$$

$$= \frac{0.1}{0.325} = \frac{100}{325} = \frac{40}{13}$$

Probability distribution -

Random variables

Discrete

dice $\rightarrow X = \{1, 2, 3, 4, 5, 6\}$

→ One value from set of values

Continuous

takes one value from range of values.

like weight $\rightarrow 0 \leq X \leq 100$ gm

Discrete distributions

(Table form)

General

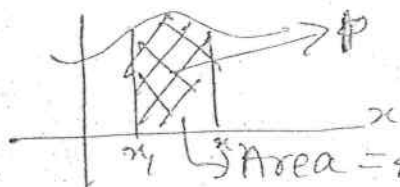
Binomial

Hypergeometric

Poisson

Continuous distri.

(Curve form)



General

Uniform

Normal, standard normal

Exponential

Discrete distributions - We get,

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

whereas in continuous prob.,

$$p(n=2) = \frac{1}{6} \rightarrow \text{discrete}$$

$$p(n=2) = 0 \rightarrow \text{Conti.}$$

1) General Distri. -

X	40	50	60	70
p(x)	0.1	0.5	0.1	0.3

$$E(x) = 40 \times 0.1 + 50 \times 0.5 + 60 \times 0.1 + 70 \times 0.3$$

$$= 56 = 4 + 25 + 6 + 21$$

In dice -

X	2	3	4	10	11	12
p(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

prob. distri. table.

it is require put all values of x.

$$1. \quad \sum p(x) = 1$$

eg.

X	3	4	5	6	7
p(x)	k	2k	3k	4k	5k

if above is prob. distri table, then k=?

Solⁿ ... $\sum p(x) = k + 2k + 3k + 4k + 5k = 1$

$$15k = 1$$

$$\boxed{k = \frac{1}{15}}$$

Ans

and $p(x=5) = \frac{3}{15} = \frac{1}{5}$

$$\mu - k\sigma = 36$$

$$56 - k \times 10 = 36$$

$$k = 2$$

$$\left. \begin{array}{l} 1 - \frac{1}{k^2} \text{ always} \\ \end{array} \right\}$$

minimum probability we can find by this rule.

$$\sum (x^2 + 1) = \sum (x^2 + 1) \cdot p(x)$$

$$= 2 \times \frac{1}{6} + 5 \times \frac{1}{6} + 10 \times \frac{1}{16} + 17 \times \frac{1}{6} + 27 \times \frac{1}{6} + 37 \times \frac{1}{6}$$

Prob. One is keep on tossing coin. What is the expected no. of toss, that he gets 2 head and stops the game?

2 - $\begin{array}{l} \uparrow \\ H H \checkmark \\ H T \times \\ T H \times \\ T T \times \end{array}$

$$\Rightarrow \begin{array}{c|c|c|c|c} X & 2 & 3 & 4 & 5 \\ \hline p(x) & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \dots \end{array}$$

3 - $\begin{array}{l} H H H \checkmark \\ H H T \checkmark \\ H T H \checkmark \\ H T T \times \\ T H H \checkmark \\ T H T \times \\ T T H \times \\ T T T \times \end{array}$

$$E(X) = \sum X \cdot p(x)$$

$$= \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \frac{1}{8} \times 4 + \dots$$

→ calculate in this manner!

$$\Rightarrow \text{let an example, } S = \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \dots$$

$$\frac{1}{2} S = \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + \dots$$

$$\Rightarrow S - \frac{1}{2} S = \frac{1}{2} S = \frac{2}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots$$

$$= \frac{1}{2} + \frac{\frac{1}{8}}{1 - \frac{1}{2}} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow S = \frac{6}{4} = \frac{3}{2} = 1.5$$

So, on avg. person will play 1.5 tosses, to get 2 heads.

Prob.

x	2	3	4	5	6	-	-	12
p(x)								

$$V(x) = \sum x^2 p(x) - (\sum x p(x))^2$$

Variance of x = ?

J.N.V

Prob.

$$E(ax+b) = a \cdot E(x) + b$$

let $E(x) = 10$, $E(2x+5) = ?$

$$\begin{aligned} \Rightarrow E(2x+5) &= 2 \cdot E(x) + 5 \\ &= 2 \times 10 + 5 \\ &= 25 \end{aligned}$$

only in case of linear funⁿ.

let if $E(ax^2+b) = \sum (ax^2+b) \cdot p(x)$
 ↗ calculate this.
 not direct formula.

let if $E(ax_1 + bx_2 + c) = a \cdot E(x_1) + b \cdot E(x_2) + c$

$E(x_1) = 5$ $\Rightarrow 3 \times 5 + 5 \times 7 + 3$
 $E(x_2) = 2$ $= 53$ \rightarrow due to

What is, $E(3x_1 + 5x_2 + 3) = ?$ linearity

$$\Rightarrow \mu_{ax+b} = a \cdot \mu_x + b$$

$$\mu_{ax_1+bx_2+c} = a \mu_{x_1} + b \mu_{x_2} + c$$

$a \rightarrow$ Scaling $b \rightarrow$ origin's shifting of origin

$$\Rightarrow \left. \begin{array}{l} \pm \Rightarrow \text{shifting} \\ * \Rightarrow \text{scaling} \end{array} \right\}$$

$$\Rightarrow V(ax+b) = a^2 \cdot V(x)$$

* Variance does not effected by shifting.

$$\Rightarrow \begin{array}{l} \sigma_{ax+b}^2 = a^2 \cdot \sigma_x^2 \\ \sigma_{ax+b} = a \sigma_x \end{array} \Rightarrow \begin{array}{l} \text{standard deviation} \\ \text{also does not effected} \\ \text{by shifting} \end{array}$$

let $E(x) = 50, V(x) = 100$

$$E(3x+5) = 3 \times 50 + 5 = 155$$

$$V(3x+5) = 3^2 \times 100 = 900$$

$$\sigma_{3x+5} = 3 \cdot \sigma_x = 30$$

$$\Rightarrow V(ax_1+bx_2+c) = a^2 V(x_1) + b^2 V(x_2) + 2ab \cdot \text{Cov}(x_1, x_2)$$

\downarrow
 Covariance

let

$$V(X_1) = 100, V(X_2) = 200, \text{Cov}(X_1, X_2) = 10$$

now find out, $V(3X_1 + 4X_2) = ?$

$$\begin{aligned} \Rightarrow V(3X_1 + 4X_2) &= 3^2 \times 100 + 4^2 \times 200 + 2 \times 3 \times 4 \times 10 \\ &= 900 + 3200 + 240 \\ &= 4340. \end{aligned}$$

$$\text{if } X_1 \text{ \& } X_2 \text{ are indep.} \Rightarrow \text{Cov}(X_1, X_2) = 0$$

$\text{Cov}(X_1, X_2)$ measures the dependency b/w X_1 and X_2 .

\Rightarrow larger value $\text{Cov}(X_1, X_2) \Rightarrow$ larger dependency.

$$\Rightarrow -\infty \leq \text{Cov}(X_1, X_2) \leq +\infty$$

* if $\text{Cov}(X_1, X_2) = +100 \rightarrow$ direct dependent
 \downarrow
both moving "in same dir"

* -ve value \Rightarrow inverse dep. \rightarrow "in opposite dir"

* 0 value \rightarrow No connection b/w them.

$$\text{Cov}(X_1, X_2) = E(XY) - E(X) \cdot E(Y)$$

$$V(X) = E(X^2) - (E(X))^2$$

$$\Rightarrow \text{Cov}(X, X) = V(X) \rightarrow \text{That's why called covariance.}$$

Two random variable problem -

Y \ X	0	1	2	$P(Y) \downarrow$
0	0.1	0.2	0.05	0.35
1	0.3	0.1	0.25	0.65
	$P(X) \rightarrow 0.4$	0.3	0.30	\rightarrow marginal prob.

Joint prob. distri table.

\Rightarrow Ques. may be ask like -

1. $P(X=1 \cap Y=0)$
2. $P(X \geq 1 \cap Y=1)$
3. $P(X=1 | Y=0)$
4. $E(X)$
5. $E(Y)$
6. $V(X)$ & $V(Y)$
7. $Cov(X, Y)$
8. $E(X | Y=1)$

\Rightarrow How to answers the above -

$$1. P(X=1 \cap Y=0) = 0.2$$

$$2. P(X \geq 1 \cap Y=1) = 0.1 + 0.25 = 0.35$$

$$3. P(X=1 | Y=0) = \frac{P(X=1 \cap Y=0)}{P(Y=0)}$$

$$= \frac{0.2}{0.35} = \frac{4}{7}$$

(uncondi \rightarrow marginal)

$$\Rightarrow \begin{array}{c|ccc} X & 0 & 1 & 2 \\ \hline P(X) & 0.4 & 0.3 & 0.3 \end{array} \quad \begin{array}{c|cc} Y & 0 & 1 \\ \hline P(Y) & 0.35 & 0.65 \end{array}$$

$$\rightarrow = 0 \times 0.1 + 1 \times 0.3 + 2 \times 0.03 = 0.3 + 0.6 = 0.9$$

$$4. E(X) = 0.9$$

$$5. E(Y) = 0.65$$

$$6. V(X) = E(X^2) - \left(\sum (X \cdot E(X))^2 \right)$$

$$= 0.69$$

$$= 1.5 - (0.9)^2$$

$$\text{conditional} = \frac{\text{Inside}}{\text{Marginal}}$$

$$7. V(Y) = 0.65 - (0.65)^2$$

$$= 0.65 - 0.4 \approx 0.25 \text{ (Approx.)}$$

$$8. \text{cov}(X, Y) = E(X, Y) - E(X) \cdot E(Y)$$

$$E(X, Y) = \sum XY \cdot P(X \cap Y)$$

$$= 1 \times 1 \times 0.1 + 2 \times 1 \times 0.25$$

$$= 0.1 + 0.5$$

$$= 0.6$$

$$\Rightarrow \text{cov}(X, Y) = 0.6 - 0.9 \times 0.65$$

$$= 0.015$$

almost indep. or dep.

we can't say this

accurately.

9.

Conditional

Give
med

When

⇒ Wh

10.

X
P(X)

9.

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y}$$

Correlation

Gives how much dependency. measures linear dependence actually.

$$= \frac{0.015}{\sqrt{0.69} \times \sqrt{0.25}}$$

$$-1 \leq \rho \leq +1$$

When ρ is close to +1 \rightarrow highly dependent
 \Rightarrow When ρ is close to -1 \rightarrow least dep.

10. $E(X|Y=1) \rightarrow$ conditional expectation

X	0	1	2
P(X)	0.4	0.3	0.3

X	0	1	2
$P(X Y=1)$	$\frac{0.3}{0.65}$	$\frac{0.1}{0.65}$	$\frac{0.25}{0.65}$

$$P(X=0|Y=1) = \frac{P(X=0 \cap Y=1)}{P(Y=1)} = \frac{0.3}{0.65}$$

$$E(X|Y=1) = \sum X \cdot P(X|Y=1)$$

$$= \frac{1 \times 0.1}{0.65} + \frac{2 \times 0.25}{0.65}$$

$$\text{if } \forall p(x \cap Y) = p(x) \cdot p(Y)$$

only then, we can conclude, that
 x & Y are independent.

if $p(x \cap Y) \neq p(x) \cdot p(Y) \Rightarrow$ Dependent,

Binomial Distribution -

n trials, x success, $p(\text{success}) = p$.

- n & p are called parameters.

- x is called random variable.

$$p(X=x) = {}^n C_x p^x (1-p)^{n-x}$$

Prob. 10 dice, 3 sixes, what prob. of success?

$$n=10, p(\text{6}) = \frac{1}{6}, X=3,$$

$$p(X=3) = {}^{10} C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$$

Prob. let 10 coins, 3 heads?

$$p(X=3) = {}^{10} C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^7$$

Variations - always; $x = 0, 1, 2, \dots$

$$p(x \geq 2) = p(x=2) + p(x=3) + \dots + p(x=6)$$

10 dice
at least two sixes

$$= 1 - p(x \leq 1)$$
$$= 1 - (p(x=0) + p(x=1))$$

$$= 1 - \left[{}^{10}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} + {}^{10}C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^9 \right]$$

$$p(x \leq 2) = p(x=0) + p(x=1) + p(x=2)$$

In binomial, $E(x) = n \cdot p$

$$V(x) = np(1-p)$$

Prob 10 dice, what expected no. of 6.

$$n = 10, \quad p(6) = \frac{1}{6}, \quad E(x) = n \cdot p$$
$$= 10 \times \frac{1}{6} = 1.66$$

\Rightarrow 1.66 of them will be 6. This is an avg. value.

$$V(x) = np(1-p)$$

$$= 10 \times \frac{1}{6} \times \frac{5}{6} = \frac{50}{36} = \frac{25}{18}$$

Standard deviation, $\sigma_x = \sqrt{\frac{50}{36}} = \frac{5\sqrt{2}}{6}$

* Dice & Coin always follows binomial distri.

Assumption for Binomial -

1. Success & failure is always there.

Prob $P(A) = 0.1$

$$P(B) = 0.2$$

$$P(C) = 0.7$$

What is prob. that out of 10, 4 will vote A.

$$\Rightarrow n=10, x=4, P(A)=0.1$$

So, using theorem,

$$= {}^{10}C_4 \cdot (0.1)^4 \cdot (0.9)^6$$

=

if prob. (Not vote for A) = ?

$$= {}^{10}C_4 \cdot (0.9)^4 \cdot (0.1)^6$$

=

2. p should be same from trial to trial.

Prob 10 cards, 3 Aces = ?

$$\Rightarrow P(3 \text{ Ace}) = {}^{10}C_3 \times \left(\frac{4}{52}\right)^3 \cdot \left(\frac{48}{52}\right)^7$$

is wrong.

But if with replacement, then it will be correct.

3. Should not be used, when we use sampling for a FINITE population WITHOUT replacement.

i.e., in infinite trials, we can use it with no problem.

4. Trial should be statistically independent, i.e. result of trial should not effect on consequent trial's result.

Prob: If $\mu = 50$, $\sigma^2 = \frac{25}{2}$, $p(X=2) = ?$

$$\Rightarrow \begin{aligned} \mu &= np \\ \sigma^2 &= np(1-p) \end{aligned}$$

Divide both eq^{ns}, $1-p = \frac{25}{50} = \frac{1}{2}$

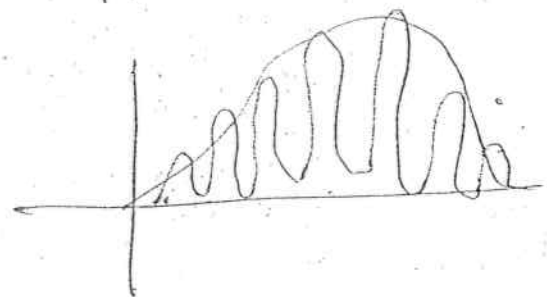
$p = \frac{1}{2}$ \rightarrow it will remain same.

$$50 = \frac{n}{2} \rightarrow n = 100$$

$$So, = {}^{100}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{98} = p(X=2)$$

Prob Binomial distri. for no. of 6 is obtained. p (no. of sixes). What is shape of bin. distri-

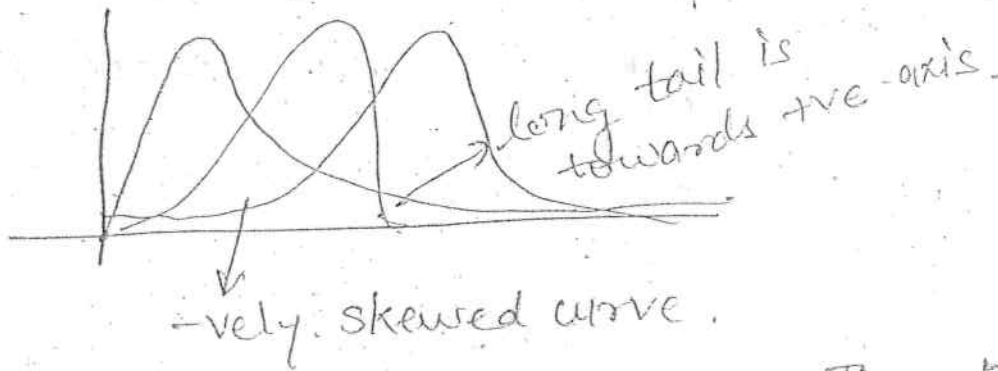
- A. Symmetric
- B. +ve skew
- C. -ve skew
- D. None.



\Rightarrow Ans. It will be symmetric shape.

Prob. In case of dice, what will shape?

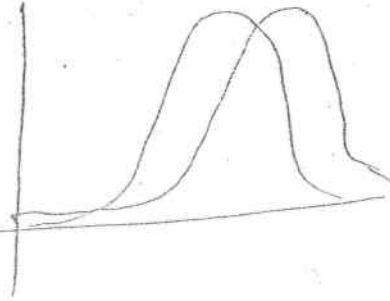
⇒ +vely skewed curve.



Prob. if getting no. ≥ 2 is success. Then prob. distri of what shape?

$$p(\geq 2) = \frac{5}{6}$$

Ans - -vely skewed.



Symmetric $\rightarrow p=q=\frac{1}{2}$

+ve skew $\rightarrow p < q, p < \frac{1}{2}, q > \frac{1}{2}$

-ve skew $\rightarrow p > \frac{1}{2}, q < \frac{1}{2}, p > q$

Remember it!

-ve : mode \geq median \geq Mean

+ve : Mean \geq ~~mode~~ Median \geq Mode

Symmetric: Mean = Med = Mode.

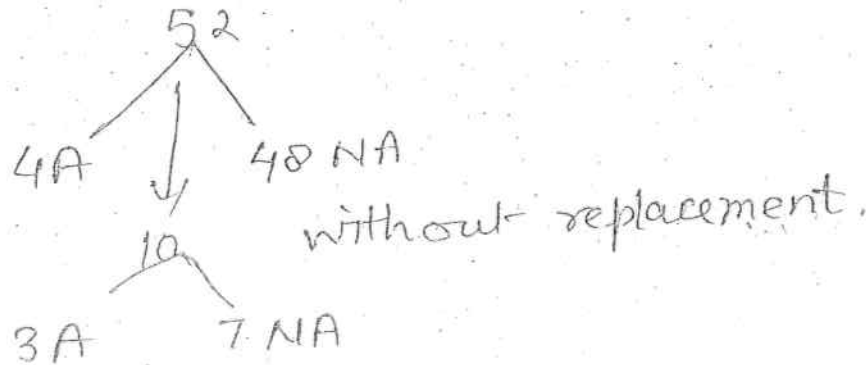
⇒ Max. freq \rightarrow Mode

$\text{Mean} = 60$, $\text{Median} = 50$, $\text{Mode} = 40$,
 \Rightarrow mode is lowest.
 Hence +vely skewed distribution.

Hypergeometric Distri-

ie; finite population without replacement.

eg - 52 cards, out of which 4 A, 48 NA.



\Rightarrow if $X=3$,

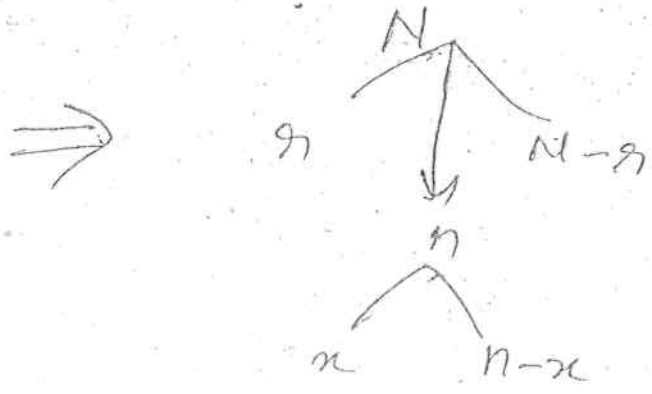
$$P(X=3) = \frac{4C_3 \times 48C_7}{52C_{10}} = \frac{n(E)}{n(S)}$$

$$P(X \geq 3) = 1 - P(X \leq 2)$$

$$= 1 - \frac{4C_0 \cdot 48C_{10} + 4C_1 \cdot 48C_9}{52C_{10}}$$

$$\left. \frac{4C_2 \cdot 48C_8}{52C_{10}} \right]$$

$$P(X=x) = \frac{{}^n C_x \cdot {}^{N-n} C_{n-x}}{{}^N C_n}$$



$$E(X) = n \cdot \frac{n}{N}$$

expected no. of success.

Poisson distri -

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}; x=0,1,2, \dots, \infty$$

- Pure poisson
- Binomial poisson.

① Case - $\lambda = \alpha \cdot \Delta t$

② case - $\lambda = np.$

λ - is avg. no. of success in observation period.

Binomial $\boxed{n, p}$ \boxed{x}

Hypergeo. $\boxed{n, n_1, N}$, \boxed{x}

Poisson $\boxed{\lambda}$ \boxed{x}

α - is avg. no. of success per unit time.
 Δt - Observation period.

* In poisson problem only, time factor is present.

Prob Let $\alpha = 40 \text{ A/hrs.}$, $\frac{1}{2} \text{ hrs.}$, $p(x=10) = p$

\Rightarrow Here, $\alpha = 40$, $\Delta t = \frac{1}{2}$

So, $\lambda = \alpha \cdot \Delta t = 40 \times \frac{1}{2} = 20$

$$\lambda = 20 \text{ A}$$

$$\text{Now, } p(x=10) = \frac{e^{-20} \cdot 20^{10}}{10!}$$

$$p(x \geq 1) = 1 - p(x=0) \\ = 1 - \frac{e^{-20} \cdot 20^0}{0!}$$

$$= 1 - e^{-20}$$

$$p(x \geq 2) = 1 - [p(x=0) + p(x=1)] \\ = 1 - \left[\frac{e^{-20} \cdot 20^0}{0!} + \frac{e^{-20} \cdot 20^1}{1!} \right] =$$

$$\left. \begin{array}{l} E(X) = \lambda \\ V(X) = \lambda \end{array} \right\} \begin{array}{l} \rightarrow \text{Here in case} \\ \text{of poisson distri.} \\ \Rightarrow \text{always holds.} \end{array}$$

Binomial poisson -

Prob Manufacturer is 10,000 tractors,
and on avg, $\frac{1}{2000}$ tractors are defective.
Then, what is prob. 4 tractors becomes
defective?

Prob if 10 tractors in year, what is $P(X=10)$
is defective

$$P(X \geq 10) = 1 - P(X \leq 9)$$

Prob $P(X=2) = ?$

$$= {}^{10,000}C_2 \left(\frac{1}{2000}\right)^2 \left(\frac{1999}{2000}\right)^{9998}$$

*Whenever n is large & p is less, then it
creates problem in calculation.

Then we approximate binomial into
poisson distri.

Then, it comes out as,

$$\Rightarrow \lambda = np.$$
$$= 10000 \times \frac{1}{2000} = 5$$

$$\text{then, } P(X=2) = \frac{e^{-5} \cdot 5^2}{2!} =$$

*When n & p ~~also~~ both large, we use

Normal distribution.

↳ continuous distri.

Recurrence Relations

Two types

Determinate order
Indeterminate order

Linear
Non-linear

LRCC
LRVC

Homogeneous
Inhomogeneous

Right
com
in

LRCC - Linear recur. constt. coeff

LRVC - " " variable coeff.

Eg -
=> B
let,
mak

Determinate order -

$$a_n = 3a_{n-1} + 4a_{n-2} + 5 \rightarrow 2^{\text{nd}} \text{ order}$$

$$a_n = 4a_{n-1} + 10 \rightarrow 1^{\text{st}} \text{ order}$$

Indeterminate order -

$$a_n = 4a_{n/2} + 5$$

i.e., order will change according to size of problem.

So, we convert 'indet.' into det. order by using

two methods -

- Exact solutions

- Master's Theorem

Then,

Eg -
=>
By -

$$a_n = 3a_{n-1}^2 + 4a_{n-2} + 5 \rightarrow \text{Non-linear}$$

$$a_n = 3a_{n-1} + 4a_{n-2} + n^2 \rightarrow \text{Linear}$$

Nonlinear can be converted into linear prob.

Similarly, $a_n = n \cdot a_{n-1}$

LRVC \rightarrow LRCC

RCC
RVC

homogeneous
inhomogeneous } $a_n + 3a_{n-1} = n^2$

Right side can be polynomial / power funⁿ or combination of both, $n^2 + n + 2$ \downarrow 4^n
in case of inhomogeneous.

Eg- $a_n - 5a_{n-1} + 6a_{n-2} = 0$
 \Rightarrow By characteristic roots method,

let, $a_n = f(n)$,
make a_{n-2} as 1. Then, eqⁿ become,

$t^2 - 5t + 6 = 0 \rightarrow$ called char. eqⁿ

\Rightarrow Solving, we get, $t = 2$ and 3 .

Then, $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$

No. of order \rightarrow No. of constt.

Eg- $a_n - 5a_{n-1} + 6a_{n-2} = 0 ; a_0 = 1, a_1 = 2$

$\Rightarrow a_0 = 1, a_1 = 2 \rightarrow$ are initial conditions.

By this, we can get value of constt.

$a_0 = C_1 \cdot 2^0 + C_2 \cdot 3^0 = 1$

$a_1 = C_1 \cdot 2^1 + C_2 \cdot 3^1 = 2$

$C_1 + C_2 = 1$

$2C_1 + 3C_2 = 2$

$C_2 = 0 \ \& \ C_1 = 1$

$\Rightarrow a_n = 1 \cdot 2^n + 0 \cdot 3^n$

$a_n = 2^n$

Ans.

When roots are same -

$$t - 4t + 2 = 0$$

$$t = 2, 2$$

$$a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n$$

$$a_0 = C_1 \cdot 2^0 + C_2 \cdot 0 \cdot 2^0 = 1$$

$$a_1 = C_1 \cdot 2^1 + C_2 \cdot 1 \cdot 2^1 = 2$$

$$\Rightarrow C_1 = 1, \quad 2C_1 + 2C_2 = 2 \Rightarrow C_2 = 0$$

if three roots are same -

$$a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n + C_3 \cdot n^2 \cdot 2^n$$

Inhomogeneous -

$$\Rightarrow a_n - 5a_{n-1} + 6a_{n-2} = 5; \quad a_0 = 1, a_1 = 2$$

$$a_n = a_n^H + a_n^P \rightarrow \text{Particular sol}^n$$

Homogeneous solⁿ

Firstly take it as homogeneous eqⁿ, i.e. put 0 at R.H.S. We will get

$$a_n^H = C_1 \cdot 2^n + C_2 \cdot 3^n$$

Now solve for particular solⁿ,

Trial: $a_n^P = d$

So, final solⁿ becomes,

	R.H.S	Trial a_n^P
	c	d
poly.	$C_0 + C_1 n$	$d_0 + d_1 n$
	$C_0 + C_1 n + C_2 n^2$	$d_0 + d_1 n + d_2 n^2$
power	$c \cdot a^n$	$d \cdot a^n$
poly x pow	$(C_0 + C_1 n) a^n$	$(d_0 + d_1 n) a^n$

like, $n \cdot 2^n \rightarrow (d_0 + d_1 n) 2^n$

- Terms of a_n^H and a_n^P should be separate completely always.

If let, $a_n^P = C_1 \cdot 1^n + C_2 \cdot 1^n$ called Double collision;

$$\Rightarrow a_n^P = d \cdot n^2$$

Since, $a_n^P = d \rightarrow$

$$d - 5d + 6d = 5$$

$$2d = 5$$

$$d = 5/2$$

$$a_n = C_1 \cdot 2^n + C_2 \cdot 3^n + 5/2$$

$$\text{So, } a_0 = C_1 + C_2 + 5/2 = 1$$

$$a_1 = 2C_1 + 3C_2 + 5/2 = 2$$

$$\text{Eg. } a_n - 5a_{n-1} + 6a_{n-2} = 3n \quad ; \quad a_0 = 1, a_1 = 2$$

Trial case, $a_n^P = d_0 + d_1 n$

$$= d_0 + d_1 n - 5(d_0 + d_1(n-1)) + 6(d_0 + d_1(n-2))$$

$$= 3n$$

$$= (d_0 - 5d_0 + 5d_1 + 6d_0 + 2d_1) - (d_1 n - 5d_1 n + 6d_1 n) = 3n$$

$$= 2d_0 + 7d_1 - 2d_1 n \quad \leftarrow \textcircled{1}$$

$$= 2d_1 = 3 \quad \leftarrow \textcircled{2}$$

$$d_1 = 3/2$$

$$2d_0 = \frac{21}{2}, \quad d_0 = \frac{21}{4}$$

$$\Rightarrow \text{So, } d_0 = c_1 \cdot 2^n + c_2 \cdot 3^n + \frac{21}{4} + \frac{31 \cdot n}{2}$$

Ans

Power function -

$$a_n - 5a_{n-1} + 6a_{n-2} = 2^n$$

$$\Rightarrow a_n = a_n^H + a_n^P$$

$$a_n^H = c_1 \cdot 2^n + c_2 \cdot 3^n$$

$$a_n^P = d \cdot 2^n$$

$$a_n - 5a_{n-1} + 6a_{n-2} = 5^n$$

$$\Rightarrow a_n^H = c_1 \cdot 2^n + c_2 \cdot 3^n$$

$$a_n^P = d \cdot 5^n$$

$$\frac{d \cdot 5^n - 5 \cdot d \cdot 5^{n-1} + 6 \cdot d \cdot 5^{n-2}}{5^{n-2}} = \frac{5^n}{5^{n-2}}$$

$$\Rightarrow 25d - 25d + 6d = 25$$

$$\Rightarrow 6d = 25$$

$$d = \frac{25}{6}$$

So, we get

$$a_n^P = \frac{25}{6} \times 5^n$$

So, complete answer

$$a_n = c_1 \cdot 2^n + c_2 \cdot 3^n + \frac{25}{6} \cdot 5^n$$

NOTE - When multiply by n , it will effect all others.

$$a_n - 5a_{n-1} + 6a_{n-2} = 5^n + 2 \cdot 7^n$$

$$\Rightarrow a_n = a_n^H + a_n^{P_1} + a_n^{P_2}$$

Now put only 5^n , then solve for d .

Then put $2 \cdot 7^n$, then solve for d .

$$\text{So, } a_n^{P_2} = d \cdot 7^n = \frac{25}{6} \cdot 5^n$$

[Only LRCC is possible] \rightarrow only possible.

LRVC \rightarrow LRCC

$$n \cdot a_n - n \cdot a_{n-1} + a_{n-2} = 5$$

$$\rightarrow n \cdot a(n) - n \cdot a(n-1) + a(n-2) = 5$$

Let $b_n = n \cdot a_n$ so, we have,

$$b_n = (n-1)a_{n-1}$$

$$\begin{aligned} n a_n - n a_{n-1} + a_{n-2} &= 0 \\ n \cdot a_n + n \cdot a_{n-1} &= 5 \end{aligned}$$

$$b_n - b_{n-1} = 5 \rightarrow \text{inhomo.}$$

$$b_n - b_{n-1} = 0 \Rightarrow t-1=0 \quad t=1$$

$$\text{So, } \frac{1}{t} = c_1 \cdot 1^0 = c_1 \Rightarrow$$

$$\boxed{b_n = c_1}$$

$$b_n - b_{n-1} = 5$$

$$b_n^p = dn = 5n$$

$$dn - d(n-1) = 5$$

$$d = 5$$

$$\text{So, } b_n = b_n^h + b_n^p \\ = c + 5n$$

$$\text{So, as } b_n = n \cdot a_n = \frac{c + 5n}{n} \quad \text{let } a_1 = 2$$

$$a_1 = \frac{c+5}{1} = 2 \quad \rightarrow \quad c = -3$$

$$a_n = \frac{-3 + 5n}{n} = 5 - \frac{3}{n}$$

\Rightarrow \therefore condition is given on a_n not on b_n .

Non-Linear \rightarrow Linear -

$$a_n^2 - 3a_{n-1}^2 = 5 \quad ; \quad a_0 = 2$$

$$\Rightarrow \text{let } b_n = a_n^2$$

$$\Rightarrow b_n - 3b_{n-1} = 5$$

$$t - 3 = 0$$

$$t = 3,$$

$$\text{So, } b_n = b_n^h + b_n^p$$

$$b_n^h = c \cdot 3^n$$

$$b_n^p = d$$

$$d - 3d = 5$$

$$d = -5/2$$

$$b_n = b_n^n + b_n^p = c \cdot 3^n - 5/2$$

As,

$$a_n = \pm \sqrt{c \cdot 3^n - 5/2}$$

∴ (+ only because initial cond. is +ve)

$$\text{So, } a_0 = \sqrt{c - 5/2} = 2$$

$$c - 5/2 = 4$$

$$c = 4 + \frac{5}{2} = \frac{13}{2}$$

$$\text{So, } a_n = \sqrt{\frac{13}{2} \cdot 3^n - \frac{5}{2}} = \sqrt{\frac{13 \cdot 3^n - 5}{2}}$$

Indeterminate order -

$$a_n = 5a_{n/3} + 7$$

$$\text{let } n = 3^k,$$

$$n/3 = 3^{k-1}$$

$$a_{3^k} = 5 \cdot a_{3^{k-1}} + 7$$

$$\text{let } b_k = a_{3^k}, \text{ so,}$$

$$b_k = 5b_{k-1} + 7$$

$$b_k - 5b_{k-1} = 7$$

$$b_k - 5b_{k-1} = 0$$

$$\frac{t-5}{t-5} = 0 \Rightarrow b_k^H = c \cdot 5^k$$

$$b_k - 5b_{k-1} = 7$$

$$b_k^p = d = -7/4$$

$$d - 5d = 7 \Rightarrow d = -\frac{7}{4}$$

$$\text{So, } b_k = c \cdot 5^k - \frac{7}{4}$$

$$a_n = c \cdot 5^k - \frac{7}{4}$$

$\therefore k = \log_3 n$, Therefore,

$$a_n = c \cdot 5^{\log_3 n} - \frac{7}{4}$$

$$\therefore a^{\log_b c} = b^{\log_c a}$$

$$\Rightarrow a_n = c \cdot n^{\log_3 5} - \frac{7}{4} ; a_1 = 5,$$

$$a_1 = c - \frac{7}{4} = 5$$

$$c = \frac{7}{4} + 5 = \frac{27}{4}$$

$$a_n = \frac{27}{4} \cdot n^{\log_3 5} - \frac{7}{4}$$

By master's theorem,

$$a_n = o(n^{\log_3 5})$$

Eq -

$$a_n = 5 \cdot a_{n/3} + n$$

$$n = 3^k$$

$$a_{3^k} = 5 a_{3^{k-1}} + 3^k$$

$$b_k = 5 \cdot b_{k-1} + 3^k$$

$$b_n - 5 b_{k-1} = 3^k$$

18.10.10

Set Theory & Algebra

Relations

- Types of relations : Closures
- Operations on relations
- Representation of rel^n
- Equivalence & Partial order rel^n ; Poset, Lattice, Boolean Algebra
- Properties of equivalence rel^n

Functions

- Types of functions : Domain & Range
- Composition of fun^n ; $f \circ g$
- Identity, inverse : f^{-1} , I

Algebra

- Semigroup
- Monoid
- Group
- Abelian gp.
- Gp. example
- Gp. properties :
 - Order of gp.
 - Order of elements
 - Cycle of gp.
 - Subgp.
 - Normal subgp.
 - Lagrange's Theorem
 - Homo & Isomorphism of gp.

Poset, Lattice & BA

1. Poset, Tiset, woset
2. Product partial order
3. Hasse diagram
4. Extreme elements of posets

- Set
 - The
 {
 ⇒
 A
 I. ϕ
 II. ϕ
 III. $\{a\}$
 IV. $\{a\}$

$A \times B$

$|A| =$

Cartesian product

5. Types of Lattice
6. Sublattice, Semilattice
7. Boolean Algebra

Relations

Lattice,
Algebra

- Set is a well defined collection of elements.
- There is no order.

$\{ \}$ → No sequence

⇒ $()$ → sequence.

$$A = \{ \phi, \{a, b\}, (1, 2), 3 \}$$

I. $\phi \in A$ — TRUE

II. $\phi \subseteq A$ — TRUE

III. $\{a, b\} \in A$ — TRUE

IV. $\{a, b\} \subseteq A \rightarrow$ FALSE $\Rightarrow \{ \{a, b\} \} \subseteq A \rightarrow$ TRUE

$$A - B = A - (A \cap B) = A \cap B^c = ab'$$

$$A \oplus B = (A - B) \cup (B - A)$$

$$= AB^c + \cancel{A}A'B$$

$$= ab' + a'b$$

$$A \times B = \{ (x, y) \mid x \in A \ \& \ y \in B \}$$

$$|A| = m, |B| = n \quad |A \times B| = mn$$

Cartesian product $A \times B \neq B \times A \Rightarrow$ Not Commutative.

$$\Rightarrow A = \{1, 2, 3\}, B = \{a, b\}$$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$\Rightarrow \boxed{R \subseteq A \times B}$$

$\{ \}$ is smallest relⁿ b/w A & B.

Biggest relⁿ is $|A| \cdot |B| = |A \times B|$

and 2^{mn} relⁿ will be there is set.

$$S = \{(1, a), (3, b)\}$$

$$\begin{array}{l} 1Sa \quad 3Sb \\ (1, a) \in S \quad (3, b) \in S \end{array}$$

Prob. $R = \{(1, a), (1, b), (2, a), (3, a)\}$

R-relⁿ set, $R(1) = ?$

$R(1) = \{a, b\}$ → i.e. all are related to 1.
 $R(2) = \{a\}$ → all are related to 2.

in equivalence relⁿ ⇒

called equivalence class.

$$[1] = \{a, b\}$$

$$[2] = \{a\}$$

Domain - 1st element } in paired sets
 Range - 2nd element }

So, Open

→ bec

Prob. let

⇒

As, R in Cardi of

$$\Rightarrow \left\{ \begin{array}{l} D(R) \subseteq A \\ \text{Range}(R) \subseteq B \end{array} \right. \quad \begin{array}{l} \text{defined in } A \times B \\ \downarrow \\ \text{Codomain} \end{array}$$

So, Codomain can be bigger than range.

Operations on relⁿ - $\cup, \cap, L^c, R-S, S-R, R \oplus S$.

$\rightarrow R \circ S, S \circ R, R^{-1}, S^{-1}$ can be done only in relⁿ because of ordered pair.

Prob. $A = \{1, 2, 3\}$

let $R = \{ (1,2) (1,3) (2,2) (3,2) \}$

$S = \{ (2,1) (2,3) (3,3) (1,1) \}$

$\Rightarrow R \cup S = \{ (1,2) (1,3) (2,2) (3,2) (2,1) \}$

$R \cap S = \{ (1,3) \}$

$R^c = \bar{R} = U - R = A \times A - R$

$= \{ (1,1) (2,1) (2,3) (3,1) (3,3) \}$

As, $R \cup \bar{R} = A \times A$

R in $|A| = n, |R| = m$, then

Cardinality of \bar{R} $\rightarrow \boxed{|\bar{R}| = n^2 - m}$

of \bar{R} $\bar{S} = \{ (1,2) (2,2) (3,1) (3,2) (3,3) \}$

$$R-S = \{ (1,2) (2,2) (3,3) \}$$

$$S-R = \{ (2,1) (2,3) (1,1) \}$$

$$R \oplus S = \{ (1,2) (2,2) (3,2) (2,1) (2,3) (1,1) \}$$

$$R \circ S = RS = \{ (x,z) \mid (x,y) \in S \ \& \ (y,z) \in R \}$$

$$= \{ (2,2) (2,3) (1,2) (1,3) \}$$

$$S \circ R = SR = \{ (1,1), (1,3) (2,1) (2,3) (3,1) (3,3) \}$$

$R \circ S \neq S \circ R$ → Not commutative

But $R \circ (S \circ T) = (R \circ S) \circ T$

$$R^{-1} \neq \bar{R}$$

$$\bar{R}^{-1} = \{ (y,x) \mid (x,y) \in R \}$$

$$\Rightarrow \bar{R}^{-1} = \{ (2,1) (3,1) (2,2) (2,3) \}$$

$$\boxed{\bar{R}^{-1} = R} \Rightarrow R \text{ is symmetric}$$

Representation of Rel^n - There are several ways

- Listing → it is finite
- Statement
- Set Builder

- Matrix rep.
- Digraph
- Arrow diagram
- Table
- Graph
- formula method.

(1,1) }
R }

3) }
ve

① Listing -
 $A = \{1, 2, 3\}$
 $R = \{(1, 2) (2, 3) (2, 2) (3, 1)\}$

② set Builder -
 $R = \{(x, y) | x \leq y\}$ on A

↳ in listing, $R = \{(1, 1) (1, 2) (1, 3) (2, 2) (3, 3) (2, 3)\}$

③ statement -

xRy iff $x \parallel y$
 like on line or planes.

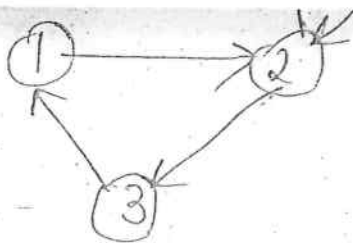
xRy iff $x \perp y$
 xRy iff $x \leq y$

ways

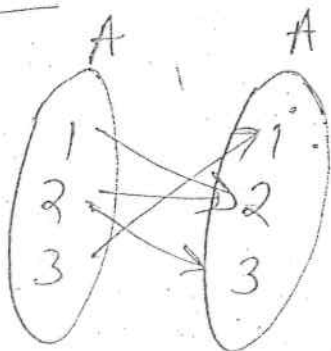
④ Matrix -

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

⑤ Digraph - Can be used only for AXA . It can't be used for AXB .



⑥ Arrow Diagram

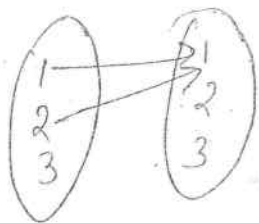


fun^n is such a rel^n where one is related to only one element.

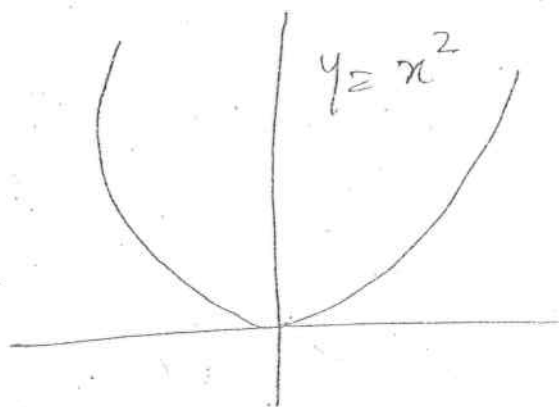
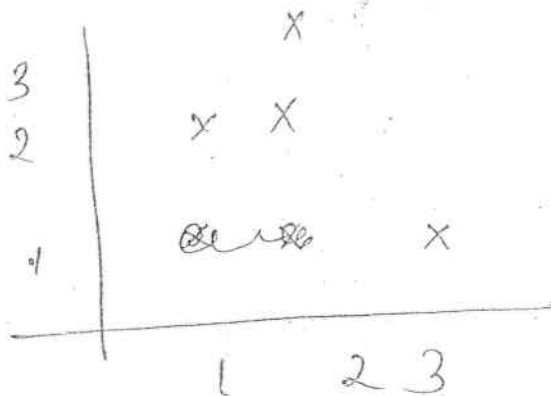
$f(x) = x^2 \rightarrow \text{fun}^n$

$(4, 2) (4, -2) \Rightarrow \text{rel}^n$

fun^n



\Rightarrow for given i/p , there should be only one o/p .



⑦ Formula method - It is particularly used for fun^n s.

Types of Rel

Reflexive	$\forall x (x, x) \in R ; \forall x, x \in X$ Selfloops
Symmetric	$(x, y) \in R \Rightarrow (y, x) \in R$ $xRy \Rightarrow yRx$
Transitive	$(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$ $xRy \wedge yRz \Rightarrow xRz$
Irreflexive	$\forall x (x, x) \notin R ; \forall x, x \in X$
Antisymmetric	$(x, y) \in R \Rightarrow (y, x) \notin R$ (unless $x=y$) $xRy \Rightarrow yRx$ i.e. self-loops are allowed.
Asymmetric	$(x, y) \in R \Rightarrow (y, x) \notin R$

$R = \{(1,1) (1,2) (3,1) (3,3)\}$ → Not Reflexive
→ Also not irreflexive

$R = \{(1,1) (2,2) (1,3) (3,3)\}$ → Reflexive

Largest reflexive set size = n^2
Minimum — — = n

$T = \{(1,3) (3,2)\}$ → Irreflexive
i.e. no self-loop should exist.

⇒

Reflexive \Rightarrow ^{Not} Irreflexive	} TRUE
Irreflexive \Rightarrow Not Reflexive	

⇒

Not reflexive \Rightarrow Irreflexive	} FALSE
Not irreflexive \Rightarrow Reflexive	

$\Rightarrow R = \{ (x, y) \mid x \leq y \}$ on $R \times R$ (by default)
 it is reflexive?

\Rightarrow This is not irreflexive but reflexive.

$$R = \{ (x, y) \mid x < y \}$$

\hookrightarrow Not reflexive but irreflexive.

$$R = \{ (x, y) \mid x + y = 10 \}$$

\hookrightarrow Neither reflexive nor irreflexive

$$R = \{ (x, y) \mid x = y^2, z \text{ is integer} \}$$

\hookrightarrow is reflexive.

$$R = \{ (x, y) \mid x \text{ is one inch from } y \}$$

(pt in plane)
 \hookrightarrow is irreflexive relⁿ.

$$R = \{ (x, y) \mid x \text{ is brother of } y \}$$

\hookrightarrow is irreflexive.

$$R = \{ (x, y) \mid x = y \}$$

\hookrightarrow is reflexive.

$$R \Rightarrow (x_1, y_1) R (x_2, y_2) \text{ iff } \begin{matrix} x_1 + y_1 = x_2 + y_2 \\ \cancel{x_1 + x_2 = y_1 + y_2} \end{matrix}$$

as, $R = \{ (1, 2), (3, 0), (1, 2), (2, 1), (1, 2), (1, 2) \}$
 is reflexive.

ault)

$\rightarrow \{(x, y) \mid x \leq y\} \rightarrow$ Not symmetric

$\rightarrow \{(x, y) \mid x + y = 10\} \rightarrow$ Symmetric

$\rightarrow \{(x, y) \mid x = y^2, z \text{ is int}\} \rightarrow$ Not symmetric

$\rightarrow \{(x, y) \mid x \text{ is brother of } y\} \rightarrow$ Not symmetric

$\rightarrow \{(x_1, y_1), (x_2, y_2) \mid x_1 + y_1 = x_2 + y_2\} \rightarrow$ Symmetric

$R = \{ \} \rightarrow$ Symmetric (By default)

* Self loops are always symmetric.

$R = \{(1, 1), (1, 2), (3, 1), (2, 2)\} \rightarrow$ Antisymmetric

\hookrightarrow Not asymmetric

$R = \{(1, 3), (3, 2)\} \rightarrow$ Antisym. & asymmetric

$xRy \ \& \ yRx \Rightarrow x=y \rightarrow$ Antisymmetric defⁿ

$xRy \ \& \ yRx = \phi \rightarrow$ Asymmetric defⁿ

$R = \{(x, y) \mid x \leq y\} \rightarrow$ Antisymmetric but not asymmetric

$R = \{(x, y) \mid x < y\} \rightarrow$ Asymmetric & antisymmetric

* Asymmetric always be Antisymmetric.

$R = \{ (1,2) (2,1) (3,1) \} \rightarrow$ Not symmetric.
 also not asymmetric.
 also not antisymmetric.

$R = \{ \} \Rightarrow$ Both symmetric & Antisymmetric
 & symmetric.

"In $\{ \}$ empty Rel" Only Reflexive property
 does not hold true."

$R = \{ (1,1) (2,2) (3,3) \} \rightarrow$ Symmetric &
 antisymmetric both.
 (largest set)

$R = \{ (x,y) \mid x+y=10 \}$

as, $x+y=10$ & $y+x=10 \nRightarrow x=y$
 so, it is ^{not anti} symmetric also
 not asymmetric.

$\{ (x,y) \mid x=y^2 \}$
 $\{ (x,y) \mid y=x^2 \} \nRightarrow x=y$
 (as $x=1$)
 \Rightarrow is antisymmetric but not
 asymmetric.

$\{ (x,y) \mid x \text{ is one inch from } y \}$
 - Not antisymmetric also
 not asymmetric.

$\{ (x, y) \mid x \text{ is brother of } y \}$
 → Not antisymmetric not asymmetric.

$\Rightarrow (x_1, y_1) R (x_2, y_2) \ \& \ (x_2, y_2) R (x_1, y_1) \Rightarrow (x_1, y_1) = (x_2, y_2)$

$x_1 + y_1 = x_2 + y_2$

$\Rightarrow R = \{ (x, y) \mid x = y \} \rightarrow$ Antisymmetric but not asymmetric.

$R = \{ (x, y) \mid x \perp y \}$

$x R y \ \& \ y R z \Rightarrow x \perp y \ \& \ y \perp z \not\Rightarrow x \perp z$

So, Not transitive.

* In checking transitive, ignore self-loop always.

$\{ (x, y) \mid x + y = 10 \} \rightarrow$ Not transitive.

$\{ (x, y) \mid x = y^2 \} \rightarrow$

$x = y^b \ \& \ y = z^a \Rightarrow x = (z^a)^b = z^{ab}$

So, it is transitive.

$\{ (x, y) \mid x < y \} \rightarrow$ Transitive.

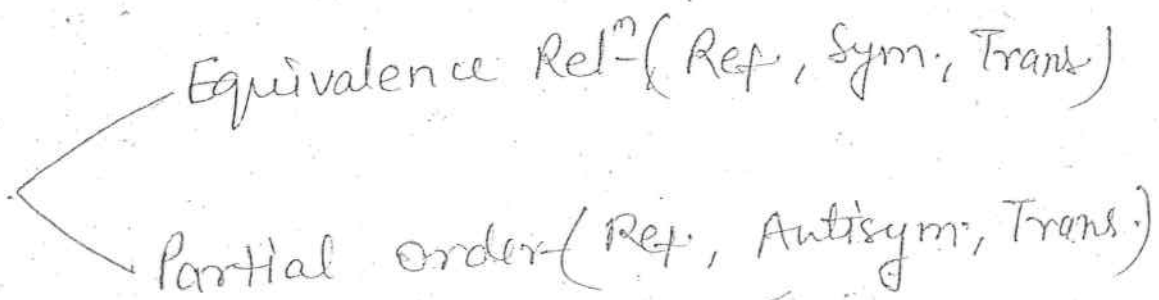
$\{ x \text{ is brother of } y \} \rightarrow$ Transitive

$\{ x_1 + y_1 = x_2 + y_2 \} \rightarrow$ Transitive.

$(x_1, y_1) R (x_2, y_2) \ \& \ (x_2, y_2) R (x_3, y_3) \Rightarrow (x_1, y_1) R (x_3, y_3)$

$x_1 + y_1 = x_2 + y_2 \ \& \ x_2 + y_2 = x_3 + y_3 \Rightarrow x_1 + y_1 = x_3 + y_3$

$(x, y) \mid x \parallel y \rightarrow$ Transitive & Reflexive.



Prob

$(x, y) \mid x \parallel y \rightarrow$
 $x < y \rightarrow$
 $x \leq y \rightarrow$

	R	S	AS	T	
$x \parallel y$	✓	✓	X	✓	Equi.
$x < y$	X	X	✓	✓	Not eq. nor Par.
$x \leq y$	✓	X	✓	✓	Partial

Word "Same" \Rightarrow Always equivalence relⁿ.

$\left\{ (x, y) \mid x \equiv y \pmod{m}, m \neq 0 \right\} \rightarrow$ Congruence modulo m relⁿ on $\mathbb{Z} \times \mathbb{R}$.

\downarrow Equivalent relⁿ because residue is same always.

So, $\left\{ (x, y) \mid x - y = 5k \right\} \rightarrow$ Integral multiple.

If only self loops \rightarrow Identity Relⁿ
 \Rightarrow Equivalence & Partial Order Both.

Closures of Rel -

- Ref. Closure
- Symm. "
- Transitive "

Reflexive Closure - Let $A = \{1, 2, 3\}$,

$$R = \{(1,1) (2,2) (1,2) (3,2)\}$$

Then ref. closure, may be defined as, "it is smallest relⁿ that contain R is reflexive."

i.e; $S = \{(1,1) (2,2) (1,2) (3,2) (3,3)\}$ is
ref. closure of R.

Symm. Closure -

$$S^1 = \{(1,1) (2,2) (1,2) (2,1) (3,2) (2,3)\}$$

Trans. Closure -

$$S^2 = \{(1,1) (2,2) (1,2) (3,2)\} \cup \{(1,2)\}$$

Prob. $R = \{(1,1) (2,2) (1,2) (3,1)\}$ $R = \{1, 2, 3, 4\}$

$$S^1 = \{(1,1) (2,2) (1,2) (3,1) (3,2) (4,3) (4,1), (4,2)\}$$

WARSHALL'S ALGO - $O(n^3)$ → 'intelligent method.'

Brute force method - $O(n^4)$

- It is used for Transitive closure finding.

R^n is symbol for trans. closure.

As,

$$R^0 = R \cup R^2 \cup R^3 \cup \dots \cup R^n$$

$$= R \cup R^2 \cup \dots \cup R^n$$

- as it is proved that beyond n , there is nothing new interesting thing happens.

Prob. $R = \{ (x,y) \mid x < y \}$ defined on $R \times R$

a. $(x,y) \mid x = y$

b. $(x,y) \mid x > y$

c. $(x,y) \mid x \leq y \rightarrow$ Ref. closure of R

d. $(x,y) \mid x \geq y$

e. None of these

f. $(x,y) \mid x \neq y \rightarrow$ Sym. closure of R

g. $(x,y) \mid x < y \rightarrow$ Trans. closure of R as it is already

Restrictions on set R -

$R \cap (B \times B) \rightarrow$ Restriction of R to set B .

Let $A = \{1, 2, 3, 4\}$

$R = \{(1,3) (2,1) (1,2) (3,3) (4,3)\} \quad B \subseteq A$

$B = \{1, 2\}$

So, $B \times B = \{(1,1) (1,2) (2,1) (2,2)\}$

$\Rightarrow R' = \{(1,1) (1,2)\} \rightarrow$ Restriction of R to set B .

Powers of Relⁿ

$R^2 = R \circ R$

$R^3 = R \circ R^2 = R \circ R \circ R = R \circ R \circ R$

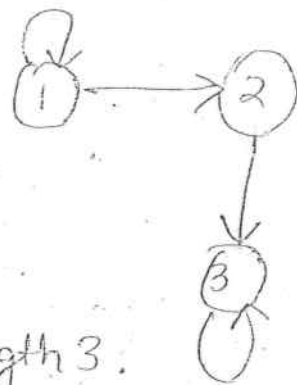
\Rightarrow Let $A = \{1, 2, 3\}$,

$R = \{(1,1) (1,2) (2,3) (3,3)\}$

$\Rightarrow R^2 = \{(1,1) (1,2) (1,3) (2,3) (3,3)\}$

if some $(x,y) \in R^2$ then

\rightarrow path of length 2.



Similarly,

R^3 contains all path of length 3.

Transitive Closure

$R^\infty = R \cup R^2 \cup \dots$

$\rightarrow (x,y) \in R^\infty$ iff \exists a path b/w x & y of any length.

Reachability relⁿ - R^n

→ It is almost same as R^∞ , except it contains self-loops also.

i.e.,

$$R^* = R^\infty \cup I$$

i.e. connected as well as self-loops.

if we have M_S & M_R , then.

$$M_{R \circ S} = M_S \circ M_R$$

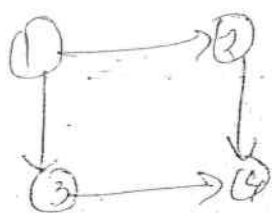
Boolean Multiplication

$$\Rightarrow M_{R^2} = M_R \circ M_R$$

if $A = \{1, 2, 3, 4\}$, and $R^5 = \begin{bmatrix} & & & 4 \\ & & & 1 \\ & & & 1 \\ 3 & & & 1 \end{bmatrix}$

then means, \exists path of length 5 from 3 to 4.

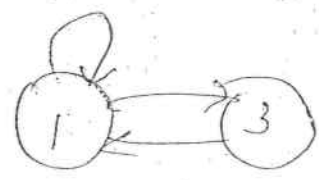
$R^5 = \begin{bmatrix} & & & 4 \\ & & & 1 \\ & & & 1 \\ 3 & & & 1 \end{bmatrix}$ gives $\Rightarrow \exists$ 6 paths of length 5 from 3 to 4.
how many paths?



$$(1, 4) \leftrightarrow 2$$

$$R^2 ?$$

self loop can be counted as path of any length

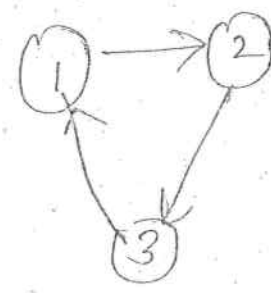


$$(3,3) \in R^{100}$$

$$\text{but } (3,3) \notin R^{101}$$

Whereas, $(1,1) \in R^{100}$ as well as $(1,1) \in R^{101}$

⇒ Let



$$\text{Now } R^{300} = \{(1,1), (2,2), (3,3)\}$$

path of length 2

$$R^{3n+2} = \{(1,3), (2,1), (3,2)\}$$

$$R^{3n+1} = \{(1,2), (2,3), (3,1)\}$$

Theorem - if R is reflexive $\Rightarrow R^{-1}$ is also reflexive.

Similarly for sym. & transitive also.

$$(x,y), (y,x) \Rightarrow (y,x), (x,y)$$

$$(x,y), (y,z), (x,z) \Rightarrow (y,x), (z,y) \Rightarrow (z,x)$$

i.e., if R is equivalence $rel^n \Rightarrow R^{-1}$ also equivalence rel^n .

- if R & S are equivalence rel^n , then
- I. $R \cap S$ also $eq^n rel^n \rightarrow$ TRUE i.e. closed under intersection
 - II. $R \cup S$ also $eq^n rel^n \rightarrow$ FALSE.

Because both contains loops, so \cap will also be contain loops.

$$\Rightarrow (x, y) \in R \cap S,$$

$$\Rightarrow (x, y) \in R \ \& \ (x, y) \in S.$$

$$\Rightarrow (y, x) \in R \ \& \ (y, x) \in S \text{ (because sym.)}$$

$$\rightarrow (y, x) \in R \cap S.$$

$$\text{let } (x, y), (y, z) \in R \cap S$$

$$\Rightarrow (x, y) \ \& \ (y, z) \in R \quad \text{--- and ---} \in S.$$

$\therefore R$ is transitive, so,

$$(x, z) \in R \quad \text{--- and } (x, z) \in S$$

$$\Rightarrow (x, z) \in R \cap S.$$

\Rightarrow If R & S are equivalence, then $R \cap S$ will also be equivalence.

But not true for $R \cup S$.

If R & S are both reflexive & symmetric, then $R \cap S$ & $R \cup S$ will also be reflexive & symmetric but it is not true in case of trans.

The
Similar
neces
Beac
tran
But

large
Sme

Theorem

\Rightarrow
 R

Now,
A/1
 $\hookrightarrow A$

- [1]
- [2]
- [3]

Theorem - If R & S are equivalent

then

$R \cap S$ is largest eq^n set.

Similarly smallest set, $R \cup S$, but it is not necessarily eq^n set.

Because $R \cup S$ may not be eq^n set, due to transitive property.

But $(R \cup S)^\circ$ is guaranteed to be smallest equivalence set.

largest eq^n set size = ~~n^2~~ n^2
smallest eq^n set size = ~~n~~ n

Theorem Every eq^n relⁿ creates a quotient set.

$\Rightarrow A = \{1, 2, 3\}$,

$R = \{ (1,1) (1,2) (2,1) (2,2) (3,3) \}$ is eq^n relⁿ.

Now,

A/R = Quotient set of A induced by R .

A/R = set of all eq^n classes of all elements of A .

$[1] = R(1) = \{1, 2\}$

$[2] = \{1, 2\}$

$[3] = \{3\}$

so, $A/R = \{ \{1, 2\}, \{3\} \}$

↳ set of distinct eq^n classes.

- A/R is always a partition of A by R .
- for every eqⁿ relⁿ \exists unique quotient set.
- for every quotient set \exists unique eqⁿ relⁿ set.

Theorem - Corresponding to any partition π of A , \exists an unique eqⁿ relⁿ R , such that,

$$A/R = \pi$$

\Rightarrow Let $\pi = \{ \{1,2\}, \{3\} \}$, $A = \{1,2,3\}$
find out eqⁿ relⁿ set.

$$\Rightarrow \pi = \{ \overline{12}, \overline{3} \} \rightarrow \text{Blocks}$$

Method - $\because \pi = \{ \{1,2\}, \{3\} \}$
 $A_1 \quad A_2$

$$R = \{ A_1 \times A_1, A_2 \times A_2 \}$$

$$= A_1 \times A_1 \cup A_2 \times A_2$$

$$R = \{ (1,1), (1,2), (2,1), (2,2), (3,3) \}$$

\Rightarrow Let $A = \{1,2,3,4\}$, $\pi = \{ \overline{12}, \overline{34} \}$

What is eqⁿ relⁿ?

$$\text{eqⁿ relⁿ } = \{ (1,1), (2,2), (1,2), (2,1), (3,3), (4,4), (3,4), (4,3) \}$$

$A/R = \{ \{1,2\}, \{3,4\} \} \rightarrow$ to check answer!

* $\left[\begin{array}{l} \text{No. of blocks at least} = 1 \\ \text{at most} = n \end{array} \right]$

Properties of eqⁿ class - Assume $A \times A$ is R .

if $[x]$ & $[y]$ are any eqⁿ classes, then

- ① $\forall x \in A, x \in [x]$ \rightarrow due to reflexivity
- ② $[x] \cap [y] = \phi$, if $[x] \neq [y]$
- ③ $\bigcup_{x \in A} [x] = A$

Properties of partition class -

Let $A = \{1,2,3\}$,

$\pi = \{ \{1,2\}, \{2,3\} \} \rightarrow$ Is not partition set.

- ① $A_i \cap A_j = \phi$, if $i \neq j$
- ② $\bigcup_i A_i = A$

Prob. $W = \{ \text{bat, ball, cat, call, catch} \}$

$R = \{ (x,y) \mid x \text{ \& } y \text{ starts with same letters} \}$

$\Rightarrow W/R = ?$

$$W/R = \{ \{ \text{bat}, \text{ball} \}, \{ \text{cat}, \text{call}, \text{catch} \} \}$$

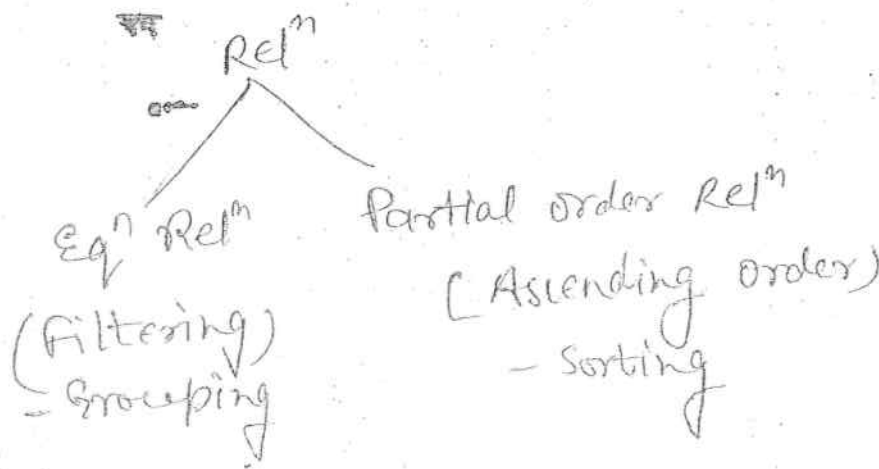
$$[\text{bat}] = \{ \text{bat}, \text{ball} \}$$

$$[\text{ball}] = \{ \text{bat}, \text{ball} \}$$

$$[\text{cat}] = \{ \text{cat}, \text{call}, \text{catch} \}$$

$$[\text{call}] = \{ \text{cat}, \text{call}, \text{catch} \}$$

$$[\text{catch}] = \{ \text{cat}, \text{call}, \text{catch} \}$$



$$\Rightarrow \text{Prob. } R = \{ (x, y) \mid |x| = |y| \}$$

$$\text{so, } W/R = \{ \{ \text{bat}, \text{cat} \}, \{ \text{ball}, \text{call} \}, \{ \text{catch} \} \}$$

$$\text{Prob. } R = \{ (x, y) \mid x \equiv y \pmod{m} \} \text{ on } \mathbb{Z}^2.$$

\Rightarrow There will be exactly m different residue.

so, there will be exact m distinct eqⁿ classes.

tech?

$$\mathbb{Z}/R = \left\{ \left\{ \pm 0, \pm 5, \pm 10, \dots \right\}, \right.$$

↓
 $y \pmod 5$

$$\left\{ \dots \pm 6, \pm 11, \pm 16, \dots \right\},$$

$$\left\{ \dots \pm 7, \pm 12, \dots \right\}, \left\{ \dots \pm 8, \pm 13, \dots \right\},$$

$$\left\{ \dots \pm 9, \pm 14, \dots \right\} \left. \right\}$$

Prob. if $x \equiv y \pmod 5$ then $[7] = ?$

$$[7] = 5x + 2 \quad \text{---}$$

$$[4] = 5x + 4 \quad \text{---}$$

functions

- Definition

- Domain & Range

- Mapping

- One to one, Many to many funⁿ

- Onto, Into funⁿ

- fog

- f^{-1}

- Identity

- Equality functions, Symmetric funⁿ

Definition - Unique valued relⁿ is called funⁿ for every I/P.

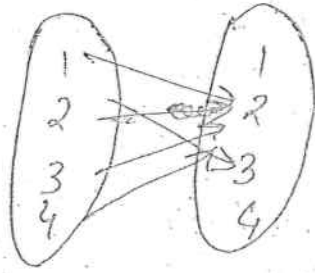
i.e. for every I/P, \exists only one O/P.

tech?

x2.

$$\Rightarrow A = \{1, 2, 3, 4\}, R = \{(1,2), (2,3), (3,2), (4,2)\}$$

↳ This is funⁿ as for every I/P \exists only one O/P.



$$f(x) = x^2$$

$$f = \{(x,y) \mid y = x^2\}$$

↳ also funⁿ

Domain & Range of funⁿ -

$$f(x) = \sqrt{x} \rightarrow \text{Not funⁿ}$$

$$f(x) = \sin x \rightarrow \text{Not funⁿ}$$

Domain -

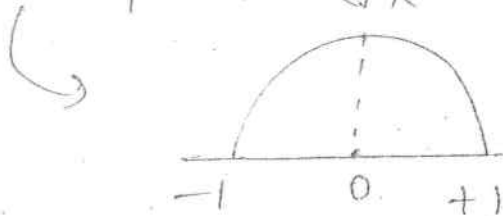
- All those I/Ps which gives meaningful O/P.

$$\Rightarrow f(x) = x^2 \text{ on } \mathbb{R} \times \mathbb{R}$$

$$\text{So, } \text{Dom}(f) = \mathbb{R}$$

$$\Rightarrow f(x) = \sqrt{1-x^2} \text{ on } \mathbb{R} \times \mathbb{R}$$

$$\text{So, } \text{Dom}(f) = \{-1 \leq x \leq 1\}$$



⇒ if Domain is equal to 1st set \Rightarrow Total funⁿ

i.e., if $A \times B$, then $\text{Dom}(f) = A \Rightarrow$ Mapping

$$f(x) = x^2 \rightarrow \text{Mapping}$$

$$f(x) = \sqrt{4-x^2} \rightarrow \text{Partial funⁿ on } \mathbb{R} \times \mathbb{R}$$

But on $A \times \mathbb{R}$, it is mapping as Total funⁿ.

- fun itself as mapping otherwise, we define its domain.

$\Rightarrow A \times B \rightarrow$ Codomain

\rightarrow Range is not necessarily ^{do} Codomain.

^{But} \Rightarrow $\text{Range}(f) \subseteq B$ always.

$\Rightarrow f(x) = x^2 ; \mathbb{R} \times \mathbb{R}$

\Rightarrow ~~Range~~ $\text{Range}(f) = \mathbb{R}^+ \cup \{0\}$

\rightarrow Nonnegative int.

$y = x^2$
 $\sqrt{y} = x$

So $\mathbb{R} =$ only +ve no. + $\{0\}$

$\Rightarrow f(x) = 3x + 1$ on $\mathbb{R} \times \mathbb{R}$

\Rightarrow Domain $(f) = \mathbb{R}$

Range $(f) = \mathbb{R}$.

Put $y = 3x + 1$

$x = \frac{y-1}{3}$ on $\mathbb{R} \times \mathbb{R}$

So, Range is always be real.

Domain of funⁿ \rightarrow Mapping \rightarrow Total

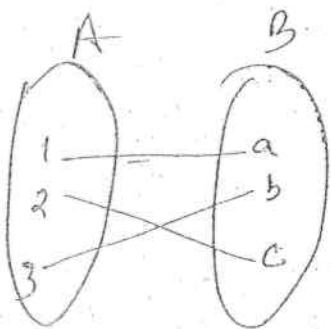
Range of funⁿ \rightarrow onto funⁿ \rightarrow Partial

Onto funⁿ - $\forall y \in B, \exists x \in A, x R y.$

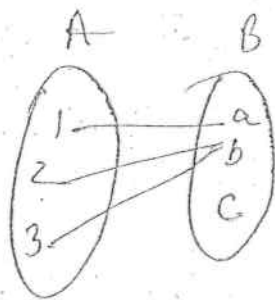
and $\text{Range}(f) = B.$

Mapping - $\forall x \in A, \exists y \in B, x R y$

Into funⁿ - Range (f) \subset B



Onto



Into

\Rightarrow ① $f(x) = e^x$ on $\mathbb{R} \times \mathbb{R}$

② $f(x) = \sin x$ on $\mathbb{R} \times \mathbb{R}$

Solⁿ
① $y = e^x \rightarrow x = \log_e y$ on $\mathbb{R} \times \mathbb{R}$

Range (f) = \mathbb{R}^+ \rightarrow Not codomain
so, not onto.

② $y = \sin x$
 $x = \sin^{-1}(y)$ $\mathbb{R} \times \mathbb{R}$

$(-1 \leq y \leq 1)$ \rightarrow Not codomain
so, not onto.

\Rightarrow $f(x) = 3x + 1$ on $\mathbb{Z} \times \mathbb{Z}$

Solⁿ - $y = 3x + 1$
 $x = \frac{y-1}{3}$ on $\mathbb{Z} \times \mathbb{Z}$

∴ For any int. y , x may get be frac.
 i.e. not integer, Hence not onto. It is "into"
 but on $\mathbb{R} \times \mathbb{R}$, it is "onto fun".

One to One - f is one to one, iff

$$\boxed{f(x_1) = f(x_2) \implies x_1 = x_2}$$

$\implies f(x) = x^2$ on $\mathbb{R} \times \mathbb{R}$

$f(x) = e^x$ - One to one

$f(x) = 3x+1$ - One to One

Solⁿ let $f(x_1) = f(x_2)$

① $x_1^2 = x_2^2 \implies x_1 = \pm x_2$ -2 \rightarrow 4
+2

Not Onto. It is Many to One.

② $e^{x_1} = e^{x_2} \implies x_1 = x_2$

\rightarrow One to one

③ $3x_1+1 = 3x_2+1 \implies x_1 = x_2$

\rightarrow One to One.

Also Onto.

$\implies f(x) = x^3$ on $\mathbb{R} \times \mathbb{R}$

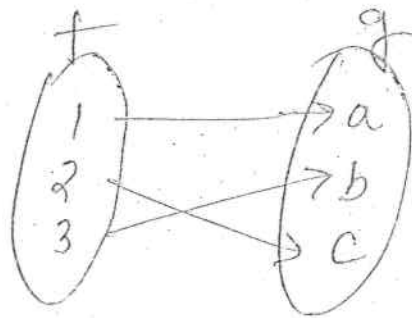
Solⁿ $x_1^3 = x_2^3 \implies x_1 = x_2, x_2\omega, x_2\omega^2$
 \Downarrow \checkmark
 complex no.

In Real, $x_1 = x_2$

So \rightarrow one to one

on $C \times C$, it will be many to one.

$f \circ g$ (Composition of f & g)



\Rightarrow One to One Correspondance funⁿ

(One to One)

if 1st element repeated \rightarrow Not-even funⁿ

\Rightarrow One to One - (Injection)
 \Rightarrow Onto (Surjective funⁿ)

Both One to One +
Onto +
Mapping } Bijeective funⁿ - Mapping is
(One to One Correspondance funⁿ)

$f \circ g$ \Rightarrow Bijeective exists b/w A & B iff
 $|A| = |B|$

Let,
 \rightarrow No bijeective b/w $N \rightarrow R$ because R is uncountable & N is countable.

→ $f = \{ (1,2) (2,3) (3,3) \}$

$g = \{ (1,3) (3,1) (3,2) \}$

$f \circ g = \{ (1,3) (3,2) (3,3) \}$
←

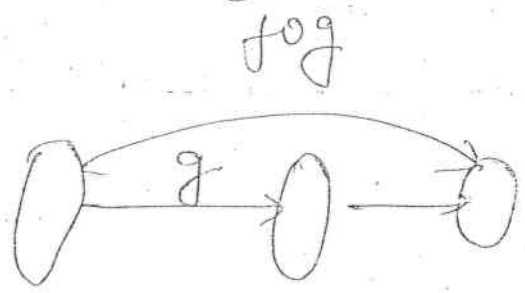
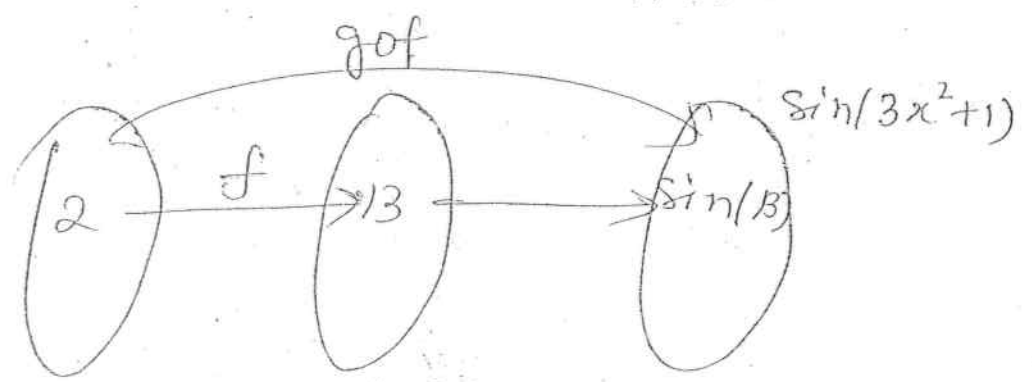
$g \circ f = \{ (2,1) (2,2) (3,1) (3,2) \}$
←

⇒ $f(x) = 3x + 1$; $g(x) = \sin x$

Solⁿ: $f \circ g(x) = f(g(x))$
 $= f(\sin x) = 3 \sin^2 x + 1$

$g \circ f(x) = g(f(x))$
 $= g(3x^2 + 1) = \sin(3x^2 + 1)$

⇒ $f \circ g \neq g \circ f$ → Not commutative
 But -
 Associative.



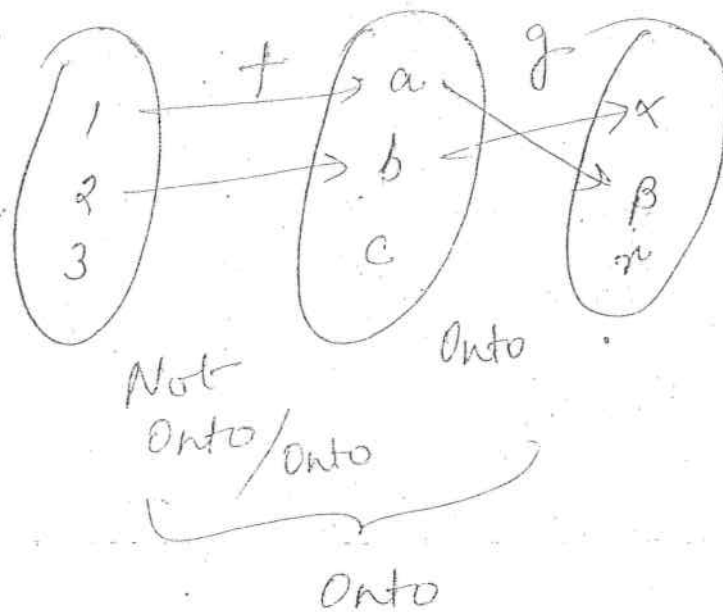
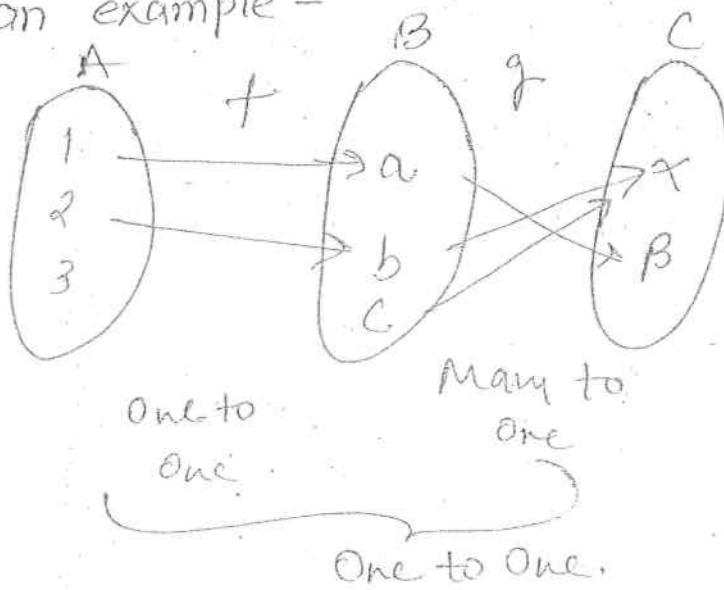
$\Rightarrow f$ is 1-1 & g is 1-1 $\Rightarrow g \circ f$ is 1-1

$\Rightarrow f$ is onto & g is onto $\Rightarrow g \circ f$ is onto

$\Rightarrow f$ is 1-1 & onto i.e. bijection & g is bijection $\Rightarrow g \circ f$ is bijection \Rightarrow

i.e., injection funⁿ & surjective funⁿ are closed under composition.

\Rightarrow Take an example -



Inverse of fun -

$$f^{-1} = \{ (y, x) \mid (x, y) \in f \}$$

into
bijection $\Rightarrow f = \{ (2, 3) (3, 5) (5, 2) \}$

solⁿ $f^{-1} = \{ (3, 2) (5, 3) (2, 5) \}$

$$\Rightarrow f(x) = 3x + 1$$

$$\therefore x = f^{-1}(y)$$

solⁿ $y = 3x + 1 \Rightarrow x = \frac{y-1}{3}$

$$\Rightarrow f^{-1}(y) = \frac{y-1}{3} \quad \swarrow \text{put } x \text{ in place of } y \text{ to get inverse.}$$

So, $f^{-1}(x) = \frac{x-1}{3}$ Ans

$$\Rightarrow f(x) = e^x$$

solⁿ $y = e^x \Rightarrow f^{-1}(y) = x = \log_e y$

$$f^{-1}(y) = \log_e y \Rightarrow \log_e x = f^{-1}(x)$$

Inverse of f .

Imp.

f is invertible, iff f^{-1} is ^{one to one} function.

$$\Rightarrow f(x) = x^2 \xrightarrow{\text{not}} \text{invertible}$$

$$= e^x \rightarrow \text{invertible.}$$

$$= 3x + 1 \rightarrow \text{invertible.}$$

* if f is bijective $\Rightarrow f^{-1}$ will also be bijective.

Eg.

if $f^{-1} = g$ & $g^{-1} = f$, then

$$\hookrightarrow \boxed{f \circ g = g \circ f = \text{Identity}}$$

Equivalence of funⁿ - Two funⁿ f & g are

equal, iff

$$\forall x \in \text{Dom}(f) \Rightarrow f(x) = g(x).$$

Eg. $f(x) = \frac{x^2 - 1}{x - 1}$, $g(x) = x + 1$

$\Rightarrow f(x) = g(x)$. as domain is same.

Prob. $f(x) = x \Rightarrow f^{-1}(x) = x = f(x)$.

$$\Rightarrow f(x) = \{(1,2)(2,1)\}$$

$$f^{-1}(x) = \{(1,2)(2,1)\} = f(x)$$

Binary operations on fun^{ns} -

$$\Rightarrow \boxed{a * b = c ; c \text{ is unique.}}$$

Also, $f(a,b) = c$

Eg. $a * b = a^2 + b^2$ on $\mathbb{R} \times \mathbb{R}$

\rightarrow it is binary funⁿ.

Bin

Can be like

the

\Rightarrow

jective.

eg. $a * b = \sqrt{ab} \rightarrow$ Not Binary operation

$$2 * 2 = \sqrt{2 * 2} = \pm 2 \Rightarrow \text{Not unique.}$$

Binary operations -

can be $a * b = a + b, ab, a^b, ab + a - b, a$ like,

$$\Rightarrow a * b = ab + a - b$$

*	a	b	c
a	b	c	a
b	a	b	a
c	a	a	b

} \Rightarrow Only useful for finite no. of elements.

if instead of b, we write d, then it will be binary opⁿ but not closed under operation *.

$$\Rightarrow \begin{aligned} a * b = a + b &\rightarrow \text{closed under } \mathbb{R} \times \mathbb{R} \\ = a - b &\rightarrow \text{Not closed under } \mathbb{N} \times \mathbb{N}. \end{aligned}$$

$$\mathbb{N} = \{0, 1, 2, \dots\}, \quad \bar{\mathbb{Z}} = \{-1, -2, \dots\}$$

$$\mathbb{Z}^+ = \{1, 2, \dots\}$$

$$\mathbb{Z} = \mathbb{Z}^+ \cup \bar{\mathbb{Z}} \cup \{0\}$$

$$= \bar{\mathbb{Z}} \cup \mathbb{N}$$

So for binary op we have to check-

- Closure
 - Associative
 - Identity
 - Inverse
 - Commutative
- } 5 properties of binary operations.

① Closure - $\forall a, b \in S, a * b \in S.$

② Associative - $\forall a, b, c \in S,$
 $a * (b * c) = (a * b) * c$

③ Identity - $\forall a \in S, e \in S$
 $a * e = e * a = a$
 $a + e = e + a = a$
(\hookrightarrow unique. It is identity for entire set.)

④ Inverse - $\forall a \in S,$
 $a * a^{-1} = a^{-1} * a = e$

also $\forall a^{-1} \in S, a^{-1}$ is unique.

⑤ Commutative - $\forall a, b \in S,$
 $a * b = b * a$

Closure Table -

*	a	b	c
a	a	b	c
b	b	a	b
c	c	b	a

identity \leftarrow

Associativity - Not so feasible to check the help of table

Identity - Row header ~~or~~ Column header, if repeated, both simultaneously.

	a	b	c
a	a	a	b
b	b	a	b
c	c	b	a

→ Not repeated row element.
→ Identity element does not exist.

Inverse - i.e., $a * a$ should be a and unique.

	a	b	c
a	a	a	b
b	a	b	c
c	b	c	a

⇒ $a^{-1} = c$
 $b^{-1} = b$
 $c^{-1} = a$

unique.
identity
entire
set.

Commutative - There should be mirror image b/w upper triangular & lower triangular matrix.

⇒ Formula Method -

$$a * b = a + b \text{ on } \mathbb{R} \times \mathbb{R}$$

$$\Rightarrow a + (b + c) = (a + b) + c$$

$$\Rightarrow a * e = e * a = a$$

$$a + e = e + a = a \Rightarrow e = 0 \rightarrow \text{Real No.}$$

$$\Rightarrow \forall a \in \mathbb{R}, a * a^{-1} = a^{-1} * a = 0$$

$$a + a^{-1} = a^{-1} + a = 0 \Rightarrow a^{-1} = -a$$

↓ $\in \mathbb{R}$

e , inverse of $a = -a$ in R .

Prob $a * b = a + b - ab$ on $R \times R$

$$\Rightarrow a + b - ab$$

$$a * (b * c) = (a * b) * c$$

$$\begin{aligned} a * (b + c - bc) &= a + (b + c - bc) - \\ &\quad a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc \end{aligned}$$

$$(a * b) * c = (a + b - ab) * c$$

$$\begin{aligned} &= ((a + b) - ab) + c - (a + b - ab)c \\ &= a + b - ab + c - ac - bc + abc \end{aligned}$$

\Rightarrow Associativity

$$a * e = e * a \quad \forall a \in R$$

$$\Rightarrow a + e - ae = e + a - ea = a$$

$$a + e - ae = a$$

$$e(1 - a) = 0$$

$$e = 0 \in R$$

$$\Rightarrow a * a^{-1} = a^{-1} * a = 0$$

$$a + a^{-1} - aa^{-1} = a^{-1} + a - a^{-1}a = 0$$

$$a + a^{-1} - aa^{-1} = 0$$

$$a^{-1}(1-a) = -a$$

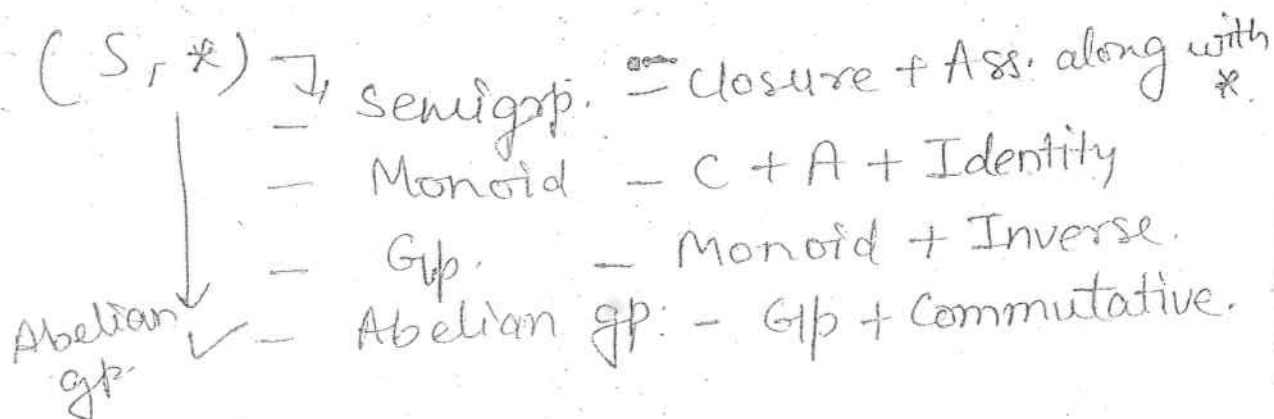
$$a^{-1} = \frac{-a}{1-a} = -\left(\frac{a}{1-a}\right)$$

So, every element has inverse except 1.

Hence it has no inverse property.

$$a + b - ab = b + a - ba$$

\Rightarrow commutative.



$(\mathbb{R}, *) \Rightarrow$ Monoid but not gp.

$(\mathbb{R} - \{0\}, *) \Rightarrow$ Abelian gp.

Group Theory

$(\mathbb{Z}, +) \Rightarrow$ Abelian gp.

$(\mathbb{Q} - \{0\}, *) \Rightarrow$ Abelian gp.

$(\mathbb{Z} - \{0\}, *) \Rightarrow$ Monoid not gp.

Basic examples of gp. -

1. $\{0, 1\}, \oplus$

id = 0,

inv. of 0, = 0,
1 = 1

⇒ Abelian gp.

\oplus	0	1
0	0	1
1	1	0

2. $(\mathbb{Z}_m, +_m)$ addition modulo m $\mathbb{Z}_m = \{0, 1, 2, \dots, m-1\}$

$0^{-1} = 0$

$1^{-1} = 3$

$2^{-1} = 2$

$3^{-1} = 1$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

gives same result, due to modulo operator.

⇒ Abelian gp.

3. $(\{1, 2, 3, \dots, p-1\}, *_p)$, prime no.

⇒ Abelian gp.

$1^{-1} = 1$

$2^{-1} = 3$

$3^{-1} = 2$

$4^{-1} = 4$

$*_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Take

$*_4$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

4 is not prime.

0 comes, hence, not closed.

4. $(S_n, 0) \rightarrow$ (Symmetric gp. of permutation)

↓
One-to-One
correspondance
funⁿ

let $S = \{1, 2, 3\}$

Now, $S_3 =$

$$P_A \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

So, S_3 will have exactly $3! = 6$ factorial fun^{ns}.

$$S_3 = \{ P_1, P_2, \dots, P_6 \}$$

So, $(S_3, 0) \rightarrow$ Gp. Not-Abelian

as composition is associative not-commutative.

⇒

	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	—	—	—	—	—
P_3	P_3	P_5	—	—	—	—
P_4	P_4	—	—	—	—	—
P_5	P_5	—	—	—	—	—
P_6	P_6	—	—	—	—	—

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = P_5$$

← ⇒ $P_2 \circ P_3 = P_5$

inverse, $p_3^{-1} = p_3$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\text{So, } p_3^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = p_3$$

$$\Rightarrow p_3^{-1} = p_3$$

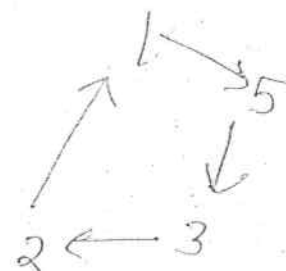
So, $(S_{n|0})$ is gp, not Abelian gp.

Theorem -

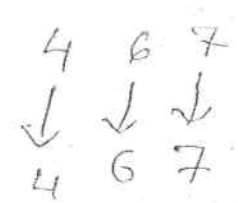
1. Every permutation funⁿ can be broken down into product of distinct disjoint cycles.
2. into product of transpositions.

① \Rightarrow let $S = \{1, 2, \dots, 7\}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 2 & 4 & 3 & 6 & 7 \end{pmatrix}$$



\Rightarrow $(1, 5, 3, 2)$ called as all cycles, remaining goes to itself



$$(3, 6, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 6 & 3 & 5 & 4 \end{pmatrix}$$

Only

\Rightarrow

\Rightarrow

- This diff

② \Rightarrow

Only if single cycle \Rightarrow Permutation cycle

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 4 & 2 & 5 & 1 & 7 \end{pmatrix} = (1,6) \circ (2,3,4) \circ \dots$$

product of disjoint cycles.

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 3 & 4 & 5 & 1 & 7 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 2 & 5 & 6 & 7 \end{pmatrix}$$

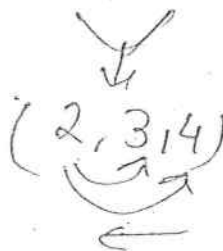
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 4 & 2 & 5 & 1 & 7 \end{pmatrix}$$

This breakdown is unique, only order may differ, and also should be in cyclic order.

② Transposition - is cycle permutation with which contains only 2 positions.

$$(x, y, z, t) = (x, t) \circ (x, z) \circ (x, y)$$

$$= (1,6) \circ (2,4) \circ (2,3)$$



$$(1,5,3,2) = (1,2) \circ (1,3) \circ (1,5)$$

If No. of transpositions in breakdown is
Even \rightarrow Even Permutation.

If No. is Odd \rightarrow Odd Permutation.

Composition of 2 Even \rightarrow Even Permut.

Comp. of Even & Odd \rightarrow Odd Permut.

Comp. of 2 Odd \rightarrow Even Permut.

Abelian gp. properties -

① A gp. $(G, *)$ is Abelian, iff,
(Two way)

$$(g * h)^2 = g^2 * h^2 \quad \forall g, h \in G$$

② If in a gp. $(G, *)$, $\forall g \in G$, and
(One way)
 $g \in g^{-1}$ then, G is abelian.

(i.e. every element has its own inverse \rightarrow Abelian)

- It is not true vice-versa.

$$a^2 = a * a$$

Proof -

$$\begin{aligned} \text{① L.H.S.} &= (g * h)^2 = (g * h) (g * h) \\ &= g * (h * g) * h \\ &= g * (g * h) * h \\ &= (g * g) * (h * h) \\ &= g^2 * h^2 = \text{R.H.S.} \end{aligned}$$

$$\Rightarrow a * b = a + b$$

$(a+b)^2 = a^2 + b^2$ is true multi gp.

$$(a+b)^2 = (a+b) + (a+b) = (a+a) + (b+b) = a^2 + b^2$$

$$\Rightarrow (R - \{0\}, *)$$

$$\hookrightarrow (a * b)^2 = a^2 * b^2$$

$$(a \times b)^2 = a^2 \times b^2$$

$$(a \times b) \times (a \times b) = (a \times a) \times (b \times b) = a^2 \times b^2$$

Proof

② \Rightarrow

\oplus	0	1
0	0	1
1	1	0

Every element is its own inverse
 \downarrow

Abelian gp.

i.e. diagonals will be identity element, e.

Abelian But converse is not true, as, $(\mathbb{Z}, +)$ is Abelian

but each element is not its own inverse.

$a * a$ Q. G is abelian iff -

a. $g = g^{-1} \quad \forall g \in G$

b. $g^2 = g \quad \forall g \in G$

c. $(goh)^2 = g^2oh^2$ ✓ is correct answer.

d. None

Abelian gp. also called as commutative gp.

Properties of Gps - (dup)

1. Order of gp.
2. Finite & infinite gp.
3. Basic properties of gps.
4. Powers of an element of a gp.
5. Order of an element of a gp.
6. Cyclic gp.
7. Subgp.
8. Normal subgps.
9. Lagrange's Theorem
10. Homomorphism & Isomorphism of gps.

Order of gp - In $G, G(G, *)$,

$$O(G) = |G|$$

$$\begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \Rightarrow |G| = 2$$

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & b & c & a \\ c & c & a & b \end{array} \Rightarrow |G| = 3$$

$$(Z, +) \Rightarrow |G| = \infty$$

Minimum order of gp = 1 \rightarrow due to identity element.

$$\begin{array}{c|c} & e \\ \hline e & e \end{array}$$

② Finite & Infinite grps -

↓
Order is finite → Infinite order.

$$(Z_m, +_m) \rightarrow \text{finite}$$

$$((1, 2, 3, \dots, p-1), \times_p) \rightarrow \text{finite}$$

$$(R, +) \rightarrow \text{Infinite}$$

③ ↓

- Id. is unique.

- Inverse is unique for given element

$$(a^{-1})^{-1} = a$$

$$(ab)^{-1} = b^{-1} * a^{-1}$$

↙ $(a*b)^{-1}$

let $(Z, +)$ → $(2 * 3)^{-1} = 3^{-1} * 2^{-1}$

2+3 ↓	3 ⁻¹ * 2 ⁻¹
5 ⁻¹ = 5	-3 + -2
	↓
	-5

④ - $ax = ay \Rightarrow x = y \rightarrow$ Left Cancellation

- $xa = ya \Rightarrow x = y \rightarrow$ Right Canc.

As, In monoid it is restricted. In (R, \times) ,

0 does not have inverse.

$$0 \times 5 = 0 \times 4$$

$$5 = 4 \quad \Downarrow$$

- $ax = b$ has unique solⁿ.

Not True

↳ $x = a^{-1} * b$ → because inverse is unique.
and in binary opⁿ solⁿ is always unique

$ya=b$ also has unique solⁿ as,

$$\boxed{y = b * a^{-1}}$$

Consequence of this, in gp. every column & row must be permutation of e .

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

→ each element is distinct.

* When

i.e. if $aa=a$
 $ab=a$ } not unique.
 → clearly not allowed in gp

Power

$b * a = b$
 $c * a = b$ } → Not allowed.

in

i.e. gp. opⁿ table can't have repetition.

* In case of distinct table, may or may not be gp.

⇒

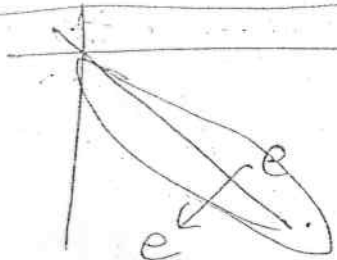
	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Q. Fill the blanks.

→ due to should not repeat

→ Gp. Opⁿ Table of finite gp. → Cayley Table.

Order



if in upper of diagonal
 ∃ any e, then its mirror
 place should also have e

⇒ O.C.

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Shift everything to LE
 ↓
 always be gp.
 Shortcut method to
 make gp.

* When matrix is symmetric in Cayley Table
 Always be gp.

Powers of an element - $\forall a \in G,$

$$a^0 = e, a^1 = a, a^2 = a * a$$

$$a^3 = a * a^2 = a^2 * a = a * a * a$$

In $(\mathbb{Z}, +) \rightarrow z^0 = 0$ $a^{-1} = \bar{a}^1$
 $z^2 = 3 + 3 = 6$ $\bar{a}^2 = a^{-1} * \bar{a}^1$

$$\bar{2}^1 = -2$$

$$\bar{2}^{-2} = -2 + -2 = -4$$

$$\bar{2}^{-3} = -2 + -2 + -2 = -6$$

$$a^m * a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

Order of gp. - $\forall a \in G,$

$\Rightarrow O(a) =$ Smallest +ve integer which satisfies
 $a^n = e$ is order of a.

\Rightarrow In $(\mathbb{Z}, +)$,
 $o(2) = \infty$ \rightarrow because not possible to find any 'sol'ⁿ,
 $2^n = 0 \rightarrow$ never get

$2^1 = 2, 2^2 = 2 + 2 = 4 \dots$

$o(0) = 1$ because, $0^n = 0 \rightarrow o^1 = 0$.

\Rightarrow $o(e) = 1$ in any gp. True always.

as, other may have any order.

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\Rightarrow o(a) = 1$

$\Rightarrow o(b) = 3$

$o(c) = 3$

$\left. \begin{matrix} o(b) = 3 \\ o(c) = 3 \end{matrix} \right\}$ b is inverse of c.

Hence, same

$b^1 = b$

$b^2 = b * b = c$

$b^3 = b * b^2 = b * c = (a) \rightarrow$ identity element

$c^1 = c$

$c^2 = c * c = b$

$c^3 = c * b = (a)$

$\Rightarrow o(a) \leq o(G)$
 ≤ 3 .

Properties of $o(a)$ -

1. $o(a) \leq o(G)$

2. $o(a) = o(a^{-1})$

3.
 4.
 let
 ⇒
 ⇒
 5.
 let
 ⇒
 like
 like
 eg. of
 Ans

sol

$$3. O(a * b) = O(b * a)$$

get

$$4. O(xax^{-1}) = O(a) ; x, a \in G$$

$$\text{let } O(a) = t, a^t = e$$

$$\Rightarrow O(xax^{-1}) = t$$

$$(xax^{-1})^t = e$$

$$= (xax^{-1}) \cdot (xax^{-1}) \cdot (xax^{-1}) \dots \dots t \text{ times}$$

$$= x(a x^{-1} x)a(x^{-1} x)a x^{-1} \dots \dots$$

$$= x a e a e a \dots \dots$$

$$= x a^t x^{-1} = x e x^{-1} = x x^{-1} = e$$

$$\Rightarrow O(x^{-1} a x) = O(a)$$

5. if $O(a) = m$, then $a^x = e$, iff m/x .

$$\text{let } O(a) = m, a^m = e$$

$\Rightarrow a^x = e \rightarrow x$ should be multiple of m .

like - $O(b) = 3, b^3 = a, b^6 = a \cdot a, b^9 = a \cdot a \cdot a$

as, $b^5 \neq a$ because 5 is not multiple of 3.

like - $b^{mn} = (b^m)^n = (e)^m = e$

eg - if $O(a) = 5$, then which can't be e -

- a. a^b
- b. a^7
- c. a^8
- d. a^{10}

Ans: a^{10} as 10 is multiple of 5.

6. $O(a) = m \Rightarrow O(a^x) = m$, if x is relatively prime to m .

if $O(a) = 5$, $O(a^3) = ?$

$O(a^3) = 5$ because 3 & 5 are relatively prime.

$O(a^7) = 5$, 5 & 7 are relatively prime.

6. Cyclic Gp. -

$\Rightarrow (G, *)$ is cyclic gp. iff $\exists a \in G$, such that

$$\forall g \in G, \quad g = a^n$$

$a =$ generator of gp.

* A cyclic gp can have more than one generator.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

- a can't be gen.

b is gen.

$$\Rightarrow b^0 = a$$

$$\Rightarrow c^0 = a \text{ also gen.}$$

$$b^1 = b$$

$$c^1 = c$$

$$b^2 = b * b = c$$

$$c^2 = c * c = b$$

* Identity can never be generators.

$\Rightarrow b$ & c both are generators in this case.

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$(\{1, 2, 3, 4\}, X_5)$
 modulo 5

$$\left. \begin{array}{l} 2^0 = 1 \\ 2^1 = 2 \\ 2^2 = 4 \\ 2^3 = 3 \end{array} \right\} 2 \text{ is gen.}$$

$$\left. \begin{array}{l} 3^0 = 1 \\ 3^1 = 3 \\ 3^2 = 4 \\ 3^3 = 2 \end{array} \right\} 3 \text{ is gen.}$$

$$\left. \begin{array}{l} 4^0 = 1 \\ 4^1 = 4 \\ 4^2 = 1 \\ 4^3 = 4 \end{array} \right\} \text{can't gen.} \\ \text{hence not-gen.}$$

\Rightarrow There are two generators 3 & 2. gen. of

① If a is gen. of cyclic gp, a^{-1} also is cyclic gp.

As, 2 & 3 are inverse of each other. Hence, they both are gen.

$$\left(\mathbb{Z}_5^+ \right) \left. \begin{array}{l} 1^{-1} = -1 \\ 1^{-2} = -2 \\ 1^2 = 1+1 = 2 \end{array} \right\} \Rightarrow \text{is cyclic gp.} \\ \text{as } 1 \text{ is gen.}$$

$$\left. \begin{array}{l} 2^0 = 1 \\ 2^1 = 2 \\ 2^2 = 4 \end{array} \right\} \text{It can't gen.}$$

\Downarrow
 So, (-1) is also gen.

That, is,

* cyclic gp. always have two generators.

$(-1)^0 = 1$ $(-1)^1 = -1$
 $(-1)^2 = 1$ $(-1)^3 = -1$
 $(-1)^4 = 1$ $(-1)^5 = -1$
 $(-1)^6 = 1$ $(-1)^7 = -1$
 $(-1)^8 = 1$ $(-1)^9 = -1$
 $(-1)^{10} = 1$ $(-1)^{11} = -1$
 $(-1)^{12} = 1$ $(-1)^{13} = -1$
 $(-1)^{14} = 1$ $(-1)^{15} = -1$

Generates gp.

$\Rightarrow (R - \{0\}, \times)$ is Not cyclic gp.

$2^1 = 2$ $3^1 = 3$ $1^1 = 1$ $0^1 = 0$ \Rightarrow No one is generating gp.
 $2^2 = 4$ $3^2 = 9$ $1^2 = 1$ $0^2 = 0$
 $2^3 = 8$ $3^3 = 27$ $1^3 = 1$ $0^3 = 0$

$\Rightarrow (R, +)$ is Not cyclic gp.

② If a finite gp. of order n , contains an element of order n , then gp. is cycle.

\Rightarrow

	a	bc
a		
b		
c		

if \exists any element of order of gp. as same size of gp., then those elements will be generators.

Prob. Gp. of 5 elements, then $O(a) = 5$
then what type of gp.
 \Rightarrow cyclic Gp.

3) Every gp. of prime order is cyclic.

$\Rightarrow 3, 5, 7, 11, \dots$ always be cyclic

4) Cyclic gp. is always Abelian Gp.

* Converse is not true.

like $(\mathbb{R}, +)$ is abelian but not cyclic gp.

$\Rightarrow \forall a, b \in G, \therefore a * b = b * a$

L.H.S, $a * b = g^m * g^n = g^{m+n} = g^{n+m}$
 $= g^n * g^m$
 $= b * a = R.H.S.$

7. Subgp.

Every subgp. of cyclic gp. is always cyclic gp.

5) The $(n, n^{\text{th}}$ roots of unity, x) is a cyclic gp.

\Rightarrow Take, $\sqrt{-1}, (\{1, -1\}, x)$

x	1	-1
1	1	-1
-1	-1	1

$(-1)^0 = 1$
 $(-1)^1 = -1$
 id = 1
 gen = -1

$\Rightarrow (\{1, \omega, \omega^2\}, x) \rightarrow$ cyclic gp.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

\Leftrightarrow Gen = ω, ω^2
 $\omega^0 = 1$
 $\omega^1 = \omega$
 $\omega^2 = \omega^2$

$(\omega^2)^0 = 1$
 $(\omega^2)^1 = \omega^2$
 $(\omega^2)^2 = \omega$

$$\Rightarrow (1, -1, i, -i)$$

$$\begin{array}{c|cccc} & 1 & -1 & i & -i \\ \hline 1 & 1 & -1 & i & -i \\ -1 & -1 & 1 & -i & i \\ i & i & -i & -1 & 1 \\ -i & -i & i & 1 & -1 \end{array}$$

Subgp - $(H, *) \in (G, *)$ iff $H \subseteq G$ and H must be a gp.

i.e., id. should be same in H & G .

$\Rightarrow (Z, +)$ has subgp. $(E, +)$

I. $(E, +) \rightarrow$ Subgp.

II. $(0, +) \rightarrow$ Not gp. because of closure prop. not closed under +.

III. $(3Z, +) \rightarrow$ Subgp.

$\hookrightarrow (kZ, +)$ also subgp, where k is integer.

$(Z, +)$ also subgp. of itself.

Always be subgp. -

I. $(G, *)$ } Trivial subgp.

II. $(\{e\}, *)$ }

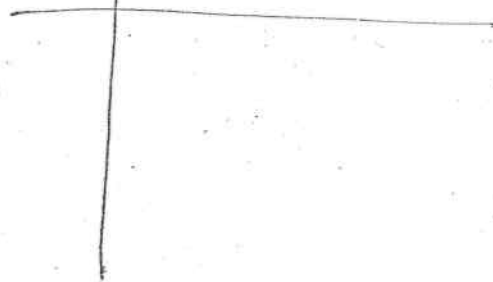
$(Z, +)$ } Trivial
 $(\{0\}, +)$ }

- other than trivial is known as Proper Subg

In case of finite sets - $\{0, 1, 2, 3, t_4\}$

t_4	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Subgp.



I. $\{0, 1, t_4\}$

II. $\{0, 2, t_4\}$

III. $\{1, 2, t_4\}$

IV. $\{0, 1, 2, t_4\}$

\rightarrow is only subgp of above
* 0 will always be present due to identity.

prop.

as I, III, IV are not closed under t_4 .

NOTE -

\Rightarrow Order of subgp divides the order of Gp.

ie. in this case for 4,

1, 2, 4 are divisible of 4.

$(\{0\}, +)$, $(\{0, 1, 2, 3\}, +)$

Trivial

inverse of 0 \rightarrow 0, 2 is 2.

$$2 + 2 = 0$$

$$0 + 0 = 0$$

No. of
 \rightarrow So, Subgroups = 3
No. of Proper subgps. = 1

Subg Subgps are closed under \cap , not under union.

Q. How many subgrp. of order of gp. 17?
 ⇒ Only 2 subgps. Due to 17 is prime no.
 Because only trivial subgps. will be there.

Normal Subgps -	$+_4$			
		0	12	3
→ 0	0	0	123	
(Right) 1	1	1	230	
→ 2	2	2	301	
3	3	3	012	

$+_4$	02
0	02
2	20

Subgp is Normal, iff $aH = Ha \quad \forall a \in G$.
 Where aH is left coset of H determined by a .
 Ha is Right coset of H determined by a .

⇒

$0H = \{0, 2\}$	$H0 = \{0, 2\}$
$1H = \{1, 3\}$	$H1 = \{1, 3\}$
$2H = \{2, 0\}$	$H2 = \{2, 0\}$
$3H = \{3, 1\}$	$H3 = \{3, 1\}$

Cosets -

⇒ $\left[\begin{array}{l} aH = \{ a * x \mid x \in H \} \\ Ha = \{ x * a \mid x \in H \} \end{array} \right]$

⇒ $\{0, 2\}$ is Normal subgp.

↳ Also proper normal subgp.

Normal subgp need not to be prop

⇒

$$\begin{array}{c|c} 4 & 0 \\ \hline 0 & 0 \end{array}$$

⇒ Also trivial Normal subgp.

$$0H = \{0\}$$

$$1H = \{1\}$$

$$2H = \{2\}$$

$$3H = \{3\}$$

Every subgp. has to be Abelian, if it is Normal subgp.

So, if any gp. is Normal, its subgp. will always be Normal subgp. Only check for Non-abelian gp.

Lagrange's Theorem - for any finite gp,

1. $|O(H)| \mid |O(G)|$

2. $|O(a)| \mid |O(G)|$

3. In any finite gp. if $|G| = n$, then $\forall a \in G$,
 $a^n = e$.

Also,

$$\Rightarrow \boxed{|O(HK)| = \frac{|O(H)| \cdot |O(K)|}{|O(H \cap K)|}}$$

H & K be any two subgp. of finite gp. G ,

$$HK = \{x \in G \mid x \in hK; h \in H, k \in K\}$$

$= O(HK)$ basically determines how many elements in HK .

Homomorphism & Isomorphism -

Homomorphism - $(G, *)$ & $(G', *_2)$

f : Mapping from $G \rightarrow G'$, defined as,
 $G \rightarrow G'$

$$\Rightarrow \boxed{f(a *_1 b) = f(a) *_2 f(b)}$$

\Rightarrow i.e., if $a \xrightarrow{G} f(a) \in G'$ & $b \xrightarrow{G} f(b) \in G'$

then, $\Rightarrow a *_1 b \rightarrow f(a) *_2 f(b)$

Let, $(\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \times)$

Now, $f(x) = e^x$ make funⁿ. Check whether homomorphism or not?

\hookrightarrow Check, $f(a *_1 b) = f(a) *_2 f(b)$

$$f(a+b) = f(a) \times f(b)$$

$$e^{a+b} = e^a \cdot e^b = e^{a+b}$$

\Rightarrow Homomorphism.

\Rightarrow Check $(\mathbb{R}^+, \times) \rightarrow (\mathbb{R}^+, \times)$

$$f(a \times b) = f(a) \times f(b)$$

$$e^{ab} = e^a \cdot e^b = e^{a+b}$$

$e^{ab} \neq e^{a+b} \Rightarrow$ Not Homomorphism

check, $(\mathbb{R}, x) \rightarrow (\mathbb{R}, x)$

$$f(n) = x^2$$

$$\Rightarrow f(a * b) = f(a) *_2 f(b)$$

$$f(a \times b) = f(a) \times f(b)$$

$$(ab)^2 = a^2 \cdot b^2$$

\Rightarrow Homomorphism.

$\Rightarrow (\mathbb{Z}, +) \rightarrow (\{1, -1\}, x)$, ~~$f(x) = x^2$~~

$$f(n) = 1 \quad ; \quad x \text{ is even}$$

$$= -1 \quad ; \quad x \text{ is odd}$$

$$f(a * b) = f(a) *_2 f(b)$$

$$f(a + b) = f(a) \times f(b)$$

a	b	a+b	f(a+b)
E	E	E	+1
E	O	O	-1
O	E	O	-1
O	O	E	+1

So,

f(a)	f(b)	f(a) * f(b)
1	1	1
1	-1	-1
-1	1	-1
-1	-1	1



Homomorphism.

Properties of Grp. Homomorphism -

$$G \rightarrow G' \\ \phi_e \rightarrow e'$$

1. $f(e) = e'$
2. $f(a^{-1}) = [f(a)]^{-1}$
3. If $S \subseteq G$, then $f(S) \subseteq G'$

Isomorphism -

1. Epimorphism \rightarrow Homo + Onto
2. Monomorphism \rightarrow Homo + One to One
3. Isomorphism \rightarrow Homo + Bijection

$$\Rightarrow (R, +) \rightarrow (R^+, \times) ; f(n) = e^n$$

Put $y = e^n \rightarrow n = \log_e y \Rightarrow$ Onto
for any +ve y , \log_e value always defined.

$$\text{let } e^{n_1} = e^{n_2} \Rightarrow n_1 = n_2 \Rightarrow \text{One to One}$$

\Rightarrow finally it is Isomorphism.

$$\Rightarrow (R^+, \times) \rightarrow (R^+, \times) , f(n) = n^2$$

Put $y = n^2 \rightarrow n = \sqrt{y} \Rightarrow$ Onto
 \rightarrow Always real.
for -ve undefined but R^+ does not contain -ve.

$$n_1^2 = n_2^2 \Rightarrow n_1 = n_2 \text{ (because does not have -ve)}$$

\Rightarrow finally Isomorphism.

$$\Rightarrow (\mathbb{Z}, +) \rightarrow (\{1, -1\}, \times)$$

$$f(n) = 1 \quad ; \text{ if } n \text{ is even} \\ = -1 \quad ; \text{ if } n \text{ is odd}$$

→ Onto funⁿ

→ Many to One because all even & odd go to ~~one~~ +ve 1 as well as -ve 1.

So, Not One to one.

⇒ This is Epimorphism.

Kernel of Homomorphism - $f: G \rightarrow G'$

$$\text{Kernel}(f) = \{x \in G \mid f(x) = e'\}$$

$$\Rightarrow (\mathbb{R}^+, \times) \rightarrow (\mathbb{R}^+, +) ; f(x) = x^2$$

$$f(x) = 0 \Rightarrow x^2 = 0 \rightarrow x = 0$$

$$\text{So, Kernel}(f) = \{0\}$$

In case of Isomorphism, x will always be Identity.

$$\Rightarrow (\mathbb{Z}, +) \rightarrow (\{1, -1\}, \times) ; f(n) = 1 \text{ , even} \\ = -1 \text{ , odd}$$

$$\text{Kernel}(f) = \{x \mid x = 2z, z \in \mathbb{Z}\}$$

Poset, Lattice & Boolean Algebra

Partial order Total order well order

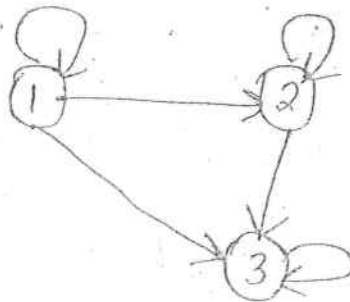
- Poset, Toset, Woset
- Hasse Diagram
- External elements of poset
- Dual poset
- Lattice
- Types of lattice
- sublattices, Semilattices
- Boolean algebra

Poset - (S, \leq)

$\Rightarrow x \leq y$ is Poset.

↓
Reflexive & Antisymm.

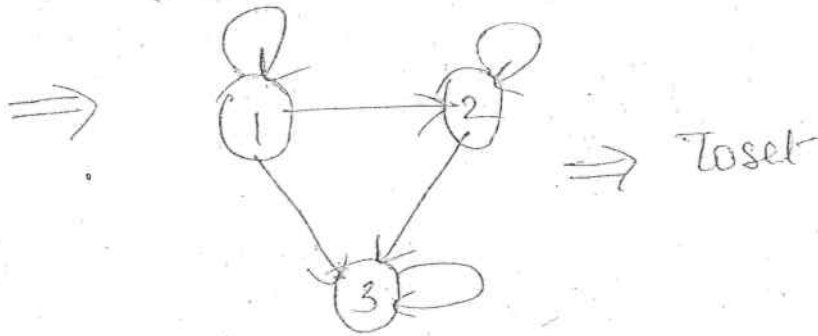
and fully transitive.



: Poset

Toset - Has to be Poset + every pair has to be comparable. $\forall x, y \in S$.

$x \leq y \text{ or } y \leq x.$



$\Rightarrow (P(S), \subseteq) \Rightarrow$ Poset

Not Toset because $\{a\} \not\subseteq \{b\}$
 $a \neq b$ called incomparable.

$\Rightarrow (R, \leq) \rightarrow$ Not Woset

$\Rightarrow (R^+, \leq) \rightarrow$ Not Woset

$\Rightarrow (1 \leq x \leq 2, \leq) \rightarrow$ Not Woset

because even if set has least element as 1, but its subset $1 < x < 2$ does not have any least element.

It is uncountably infinite set.

So, check \rightarrow $e^{\mathbb{N}}$ set
(poset + comparable) \neq

Toset + Least Element + Discrete sets \Rightarrow Woset.

Real, Complex, Rational \rightarrow does not have least elements.

$\Rightarrow (D_n, |)$

$D_n = \{ \text{all +ve integral divisors of any int } n \}$
 $n \in \mathbb{Z}$

$D_{20} = (\{1, 2, 4, 5, 10, 20\}, |) \rightarrow$ Not Toset

$\Rightarrow ((1, 2, 20), |) \rightarrow$ Woset

$(\mathbb{Z}, \geq) \Rightarrow$ Woset

least \rightarrow does not mean smallest no.
it is like starting pt.



$(\mathbb{Z}, \leq) \Rightarrow$ Not Woset, because

least element = $-\infty$

Hasse Diagram -

Product Partial Order -

$$(x_1, y_1) R (x_2, y_2) \text{ iff } x_1 \leq x_2 \ \& \ y_1 \leq y_2$$

$$\Rightarrow (1, 2) R (2, 3) \quad \checkmark$$

$$\Rightarrow (1, 2) R (0, 1) \quad \times$$

$$(x_1, y_1) R (x_2, y_2) \ \& \ (x_2, y_2) R (x_1, y_1) \Rightarrow (x_1, y_1) R (x_2, y_2)$$

Anti-Symm: $x_1 \leq x_2$
 $y_1 \leq y_2$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ x_2 \leq x_1 & & x_1 = x_2 \\ y_2 \leq y_1 & \Rightarrow & y_1 = y_2 \end{matrix}$$

$$(x_1, y_1) R (x_2, y_2) \ \& \ (x_2, y_2) R (x_3, y_3) \Rightarrow (x_1, y_1) R (x_3, y_3)$$

Transitivity

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ x_1 \leq x_2 & x_2 \leq x_3 & x_1 \leq x_3 \\ y_1 \leq y_2 & y_2 \leq y_3 & y_1 \leq y_3 \end{matrix}$$

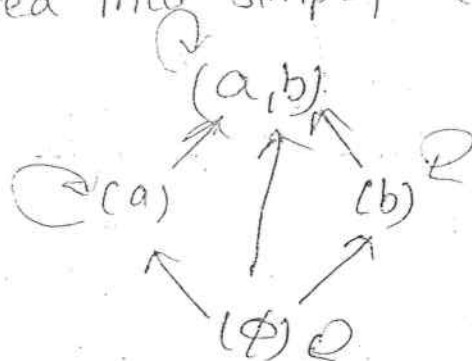
AND \rightarrow Only then holds transitivity
 OR \rightarrow No Transitivity will hold.

\Rightarrow This is Poset, not Toset.

Hasse Diagram - Diagram of Poset can be reduced into simplified form, called

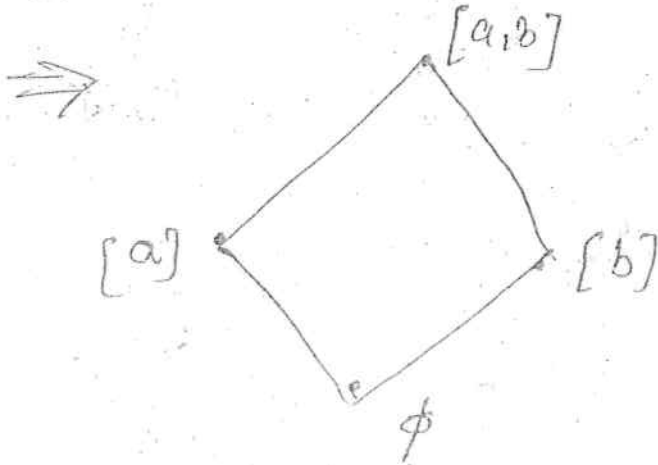
Hasse diagram.

$$\Rightarrow (P(S), \subseteq)$$



Steps -

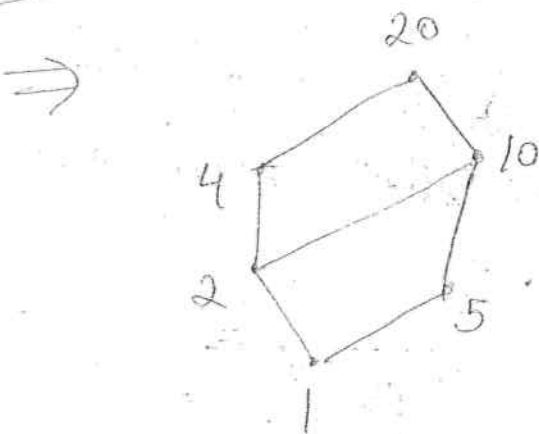
1. $\circ \longrightarrow \cdot$
2. Remove \curvearrowright loops.
3. All arrows pointing upwards.
remove head of arrow \wedge .
4. Remove all transitive arrows.



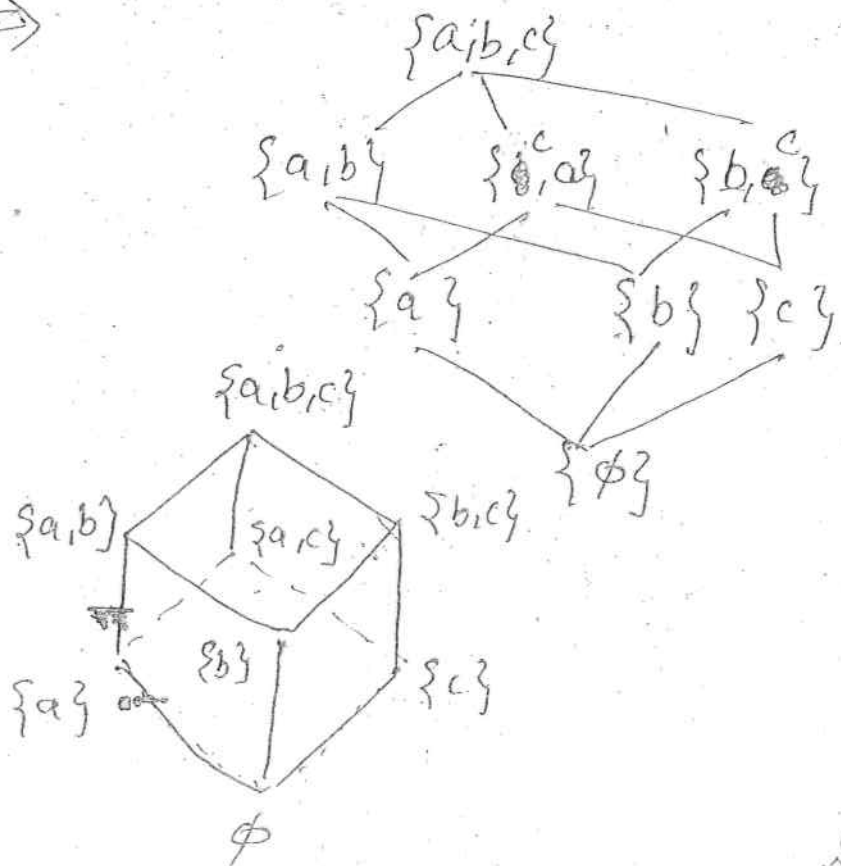
Prob. $S = \{a\}$, $P(S) = \{\phi, \{a\}\}$ $(P(S), \subseteq)$

\Rightarrow Hasse Diagram,

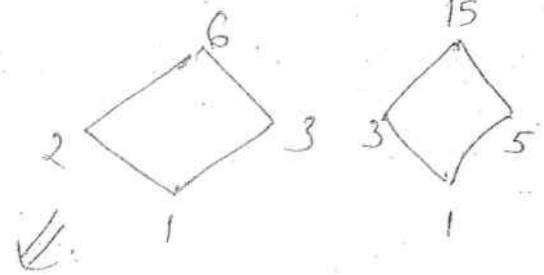
Prob. Draw H-Diagram $D_{20} = \{1, 2, 4, 5, 10, 20, 1\}$



⇒



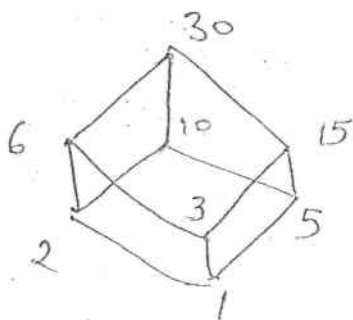
⇒ $D_6 = (\{1, 2, 3, 6\}, 1)$



called as

Atomomorphic H-Diagram.

⇒ $D_{30} = (\{1, 2, 3, 5, 6, 10, 15, 30\}, 1)$



⇒ $D_{20} = (\{1, 2, 4, 5, 10, 20\}, 1)$

* $D_{15} \& D_6 \Rightarrow$ Isomorphic Diagram.

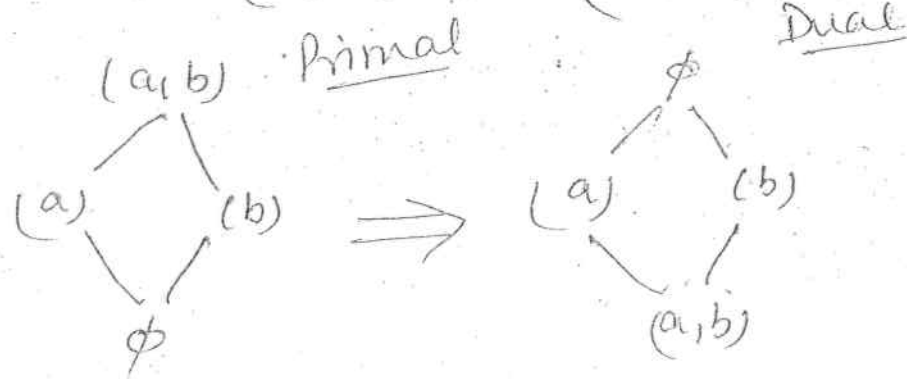
Prob: How many edge in D_{30} ?

\Rightarrow 12 edges; as due to cube structure.

Dual Poset - Inverse Rel^n exists.

like, if $(S, \subseteq) \rightarrow$ dual poset (S, \supseteq)

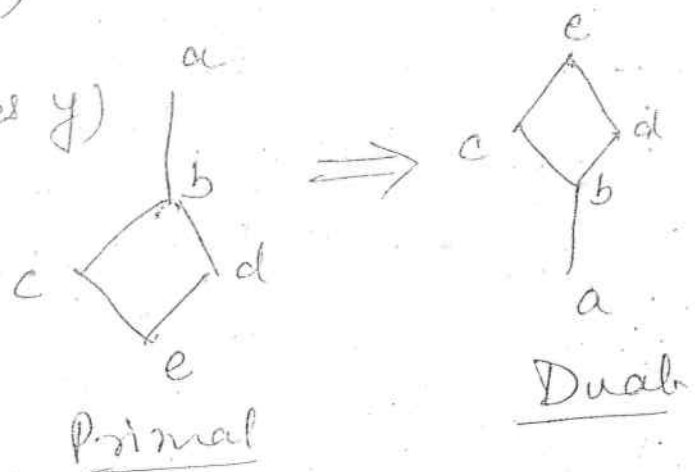
$(S, \subseteq) \rightarrow (S, \supseteq)$



* Dual of dual \Rightarrow Primal Poset.

$\Rightarrow (D_n, |) \rightarrow$ dual $(D_n, n \text{ is divisible by } y)$

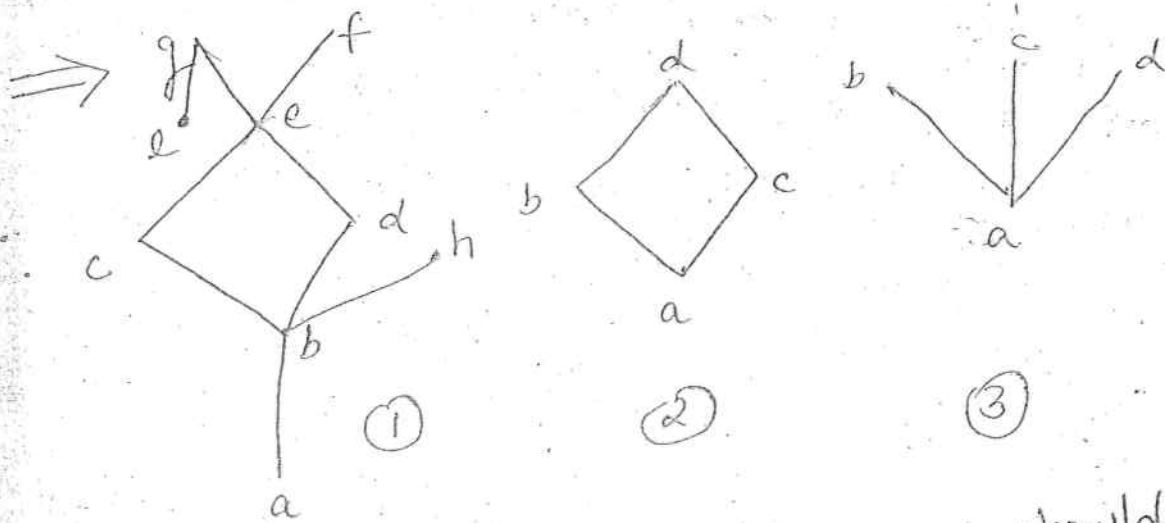
$(n \text{ divides } y)$



External elements of Posets -

1. Find maximal elements
2. " minimal "
3. greatest elements
4. least "
5. Upper Bounds

- 6. Lower Bounds
- 7. LUB (Least Upper Bound)
- 8. GLB (Greatest Lower Bound)



Maximal \rightarrow Above which no element should there.

i.e., $x \in S$ maximal iff $\nexists y \in S \mid x < y$

- ① $\rightarrow \{g, f, h\}$
 - ② $\rightarrow \{d\}$
 - ③ $\rightarrow \{b, c, d\}$
- } maximal

Minimal \rightarrow Nothing below. no

i.e., $x \in S$ minimal iff $\nexists y \in S \mid y < x$

- ①, ②, ③ $\rightarrow \{a\} \rightarrow$ minimal
- ① $\rightarrow \{a, e\} \rightarrow$ minimal

Greatest \rightarrow Only one unique element, may not exist.

i.e., $x \in S$ greatest iff $\forall y \in S \mid y \leq x$

- Greatest always be from one of maximal.
- ① \rightarrow No greatest-element
 - ② $\rightarrow d$
 - ③ \rightarrow No.

Least \rightarrow

$x \in S$ least iff $\forall y \in S \mid x \leq y$

if we have several minimal or maximum then we would never get least or greatest elements.

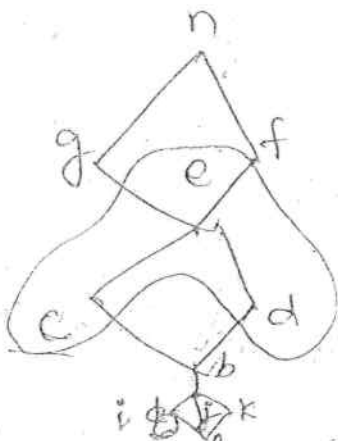
- ① \rightarrow No element
 - ② $\rightarrow a$
 - ③ $\rightarrow a$
- } least

* In Boolean, least & greatest are defined by 0 and 1.

Upper Bound - let S , and $A \subseteq S$, Then

$UB(A) = \{ x \text{ is } UBL(A) \text{ iff } \forall y \in A \mid y \leq x \}$

① $A = \{ c, d, e \}$, $U(A) = \{ e, g, f, n \}$



$L(A) = \{ b, a, i, j, k, l \}$

Lower Bound - x is $LBL(A)$ iff $\forall y \in A \mid x \leq y$

* LUB & GLB are always unique.

Least upper Bound -

$x \in S$ is LUB of $A \subseteq S$, iff ^{where}

- ① $x \in UB(A)$
- ② $\forall c \in UB(A), x \leq c$

* Nearest one.

Greatest Lower Bound - $x \in S$ in $GLB(A)$, iff

- ① $x \in LBC(A)$
- ② $\forall c \in LBC(A)$.

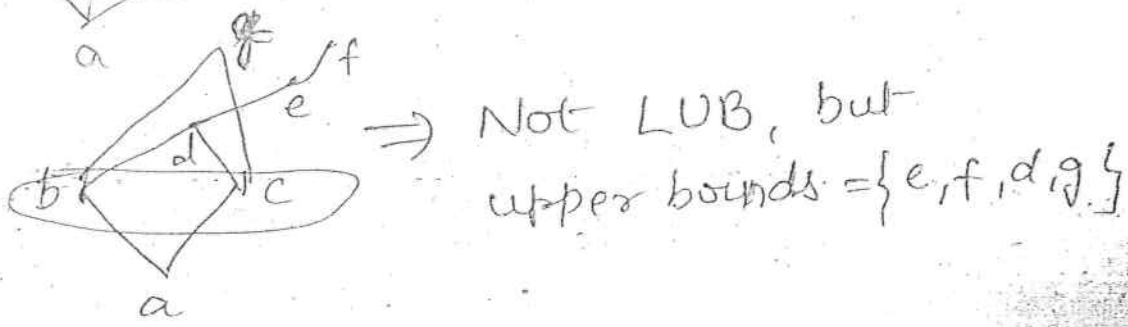
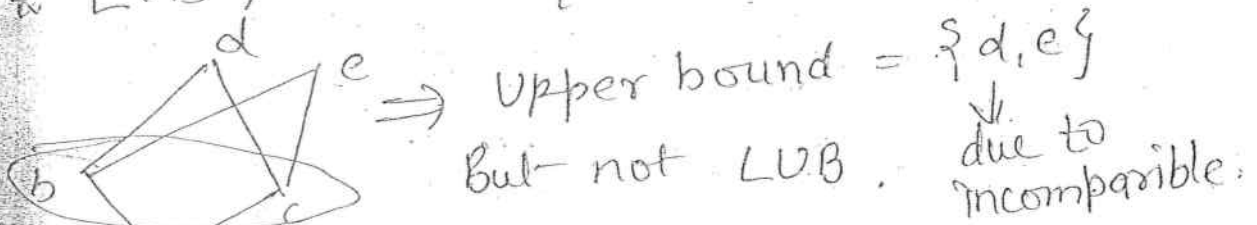
$(c, d) \rightarrow LU = e$
 $GL = b$

* First meeting pt. always be $LU \neq GL$

bounds. [Upward - LUB
 Downward - GLB.]

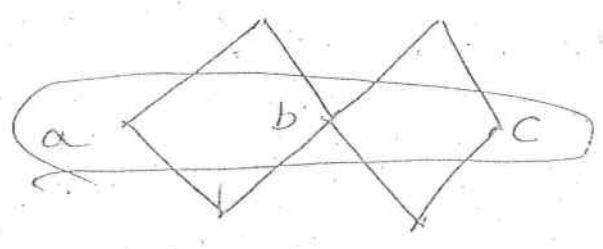
$\{b, c, e, g\} \Rightarrow LU = g$
 $GL = b$

* LUB \neq GLB may not exist.



If we can compare any two lower bounds, there, greatest lower bound will not exist.

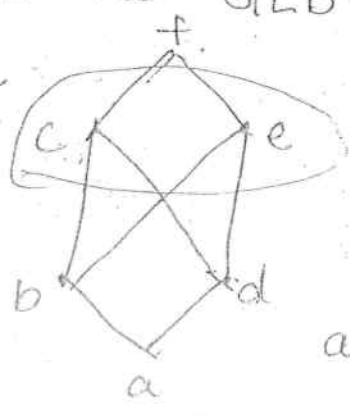
Prob.



$LUB(a, b, c) \rightarrow$ No LUB.

also no GLB of this.

Prob.

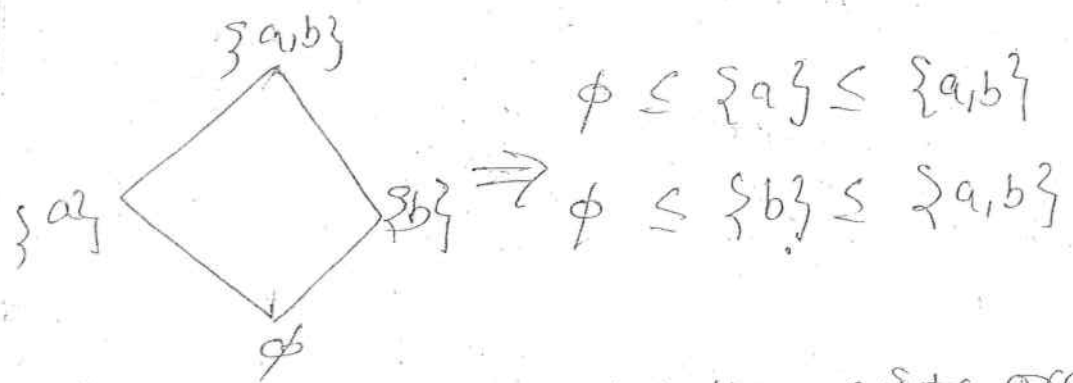


$LUB = f$

GLB = does not exist.

as, 'a' can't be GLB, because we have to consider only nearest one.

Topological Sorting - By which Poset can be converted into Tosest.



convert it without violating exists order.
 Also called compatible Tosest \rightarrow Topological Sorting of Poset.

- $\emptyset \leq \{a\} \leq \{a,b\} \leq \{b\}$ X
- $\emptyset \leq \{b\} \leq \{a\} \leq \{a,b\}$ ✓
- $\emptyset \leq \{a\} \leq \{b\} \leq \{a,b\}$ ✓
- $\{a\} \leq \{b\} \leq \emptyset \leq \{a,b\}$ X

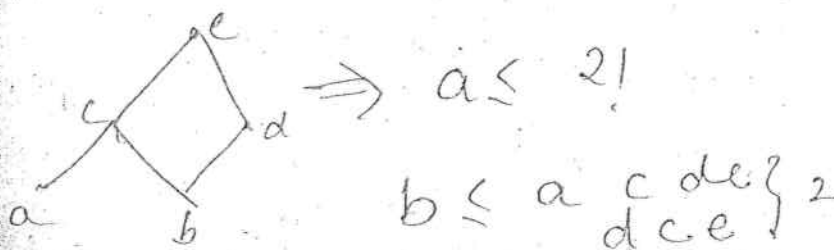
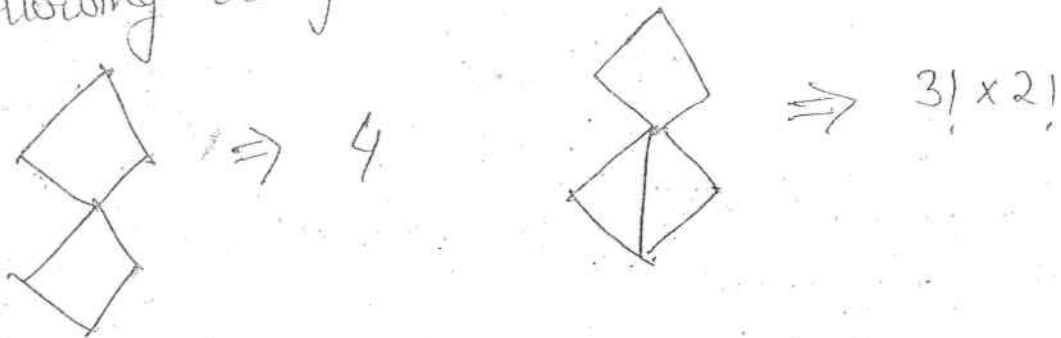
Steps of sorting -

- ① Select a minimal element
- ② Delete that element.
- ③ Goto step ①.

So, we get, $\emptyset \leq \{b\} \leq \{a\} \leq \{a,b\}$.

or $\emptyset \leq \{a\} \leq \{b\} \leq \{a,b\}$.

Prob: How many Topological sorting exists in following diagram -



$b \leq a \ c \ d \ e \} 2$
 $d \ c \ e \} 2$

$b \leq d \leq a \ c \ e \} 1$

$\Rightarrow 2 + 2 + 1 = 5$ Topologic order.

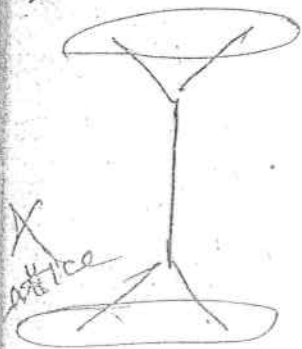
now using ~~trans~~, antisymm;

$$a_1 \leq a_1 \leq \dots \leq a_1 \Rightarrow \text{Self Loops only.}$$

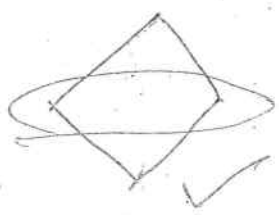
Lattice - A Poset is Lattice, iff

$\forall a, b \in S, \text{LUB}(a, b) \ \& \ \text{GLB}(a, b)$
must exist and should belong to S.

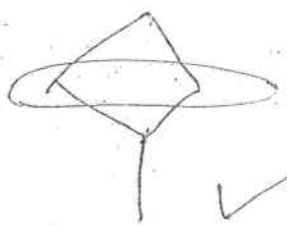
Prob Which of poset is lattice?



X
Lattice



Lattice



Lattice

* Test only for incomparable Diagram.

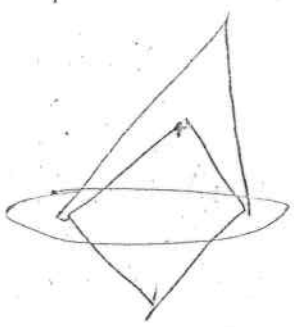
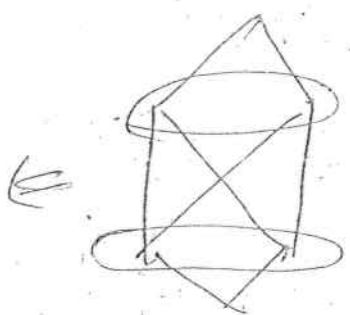
* Open at any side will never be lattice.

* But it is not necessary that closed will be lattice, we have example,

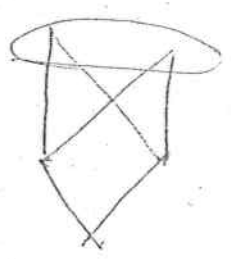
ave
one,

enty

No
LUBs
&
GLB.



Not
Lattice



Not
Lattice



Lattice



Lattice

Pentagon Lattice

Properties of Lattice \Rightarrow (does not have distrib prop)

1. Commutative
2. Associative
3. Idempotent
4. Law of Absorption
5. Closure

$$LUB(a,b) = a \vee b \Rightarrow OR \rightarrow \text{Join}(a,b)$$

$$GLB(a,b) = a \wedge b \Rightarrow AND \rightarrow \text{Meet}(a,b)$$

LUB = Supremum (a,b) = Union

GLB = Infimum (a,b) = Intersection

① Commutative - $a \vee b = b \vee a$; $\forall a,b \in S$
 $a \wedge b = b \wedge a$

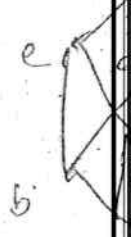
② Associative - $a \vee (b \vee c) = (a \vee b) \vee c$; $\forall a,b,c \in S$
 $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;

③ Idempotent - $a \vee a = a$
 $a \wedge a = a$; $\forall a \in S$

④ Law of absorption - $a \vee (a \wedge b) = a$; $\forall a,b \in S$
 $a \wedge (a \vee b) = a$

Need not be exists some properties -

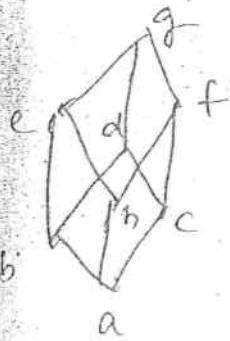
- Bounded
- Distributed
- Complemented } \rightarrow if we add these prop, lattice becomes Boolean Algebra.



$$* \left[a \leq b; \text{ iff } a \vee b = b \ \& \ a \wedge b = a \right]$$

$$\Rightarrow \left. \begin{aligned} a \leq a \vee b \ \& \ b \leq a \vee b \\ a \wedge b \leq a \ \& \ a \wedge b \leq b. \end{aligned} \right\}$$

$$\Rightarrow \text{if } a \leq b \ \& \ c \leq d \Rightarrow a \vee c \leq b \vee d. \\ a \wedge c \leq b \wedge d;$$



* Complemented is only possible if set is bounded.

So, Complemented + Distributed \Rightarrow Boolean Algebra.

in lattice $\rightarrow (S, +, \cdot)$

$$\left(S, +, \cdot, ' \right)$$

\downarrow \downarrow \downarrow
 \vee \wedge \neg

6 properties of Boolean Algebra -

- Closure
- Comm.
- Assoc.
- Distri
- Identity
- Complement

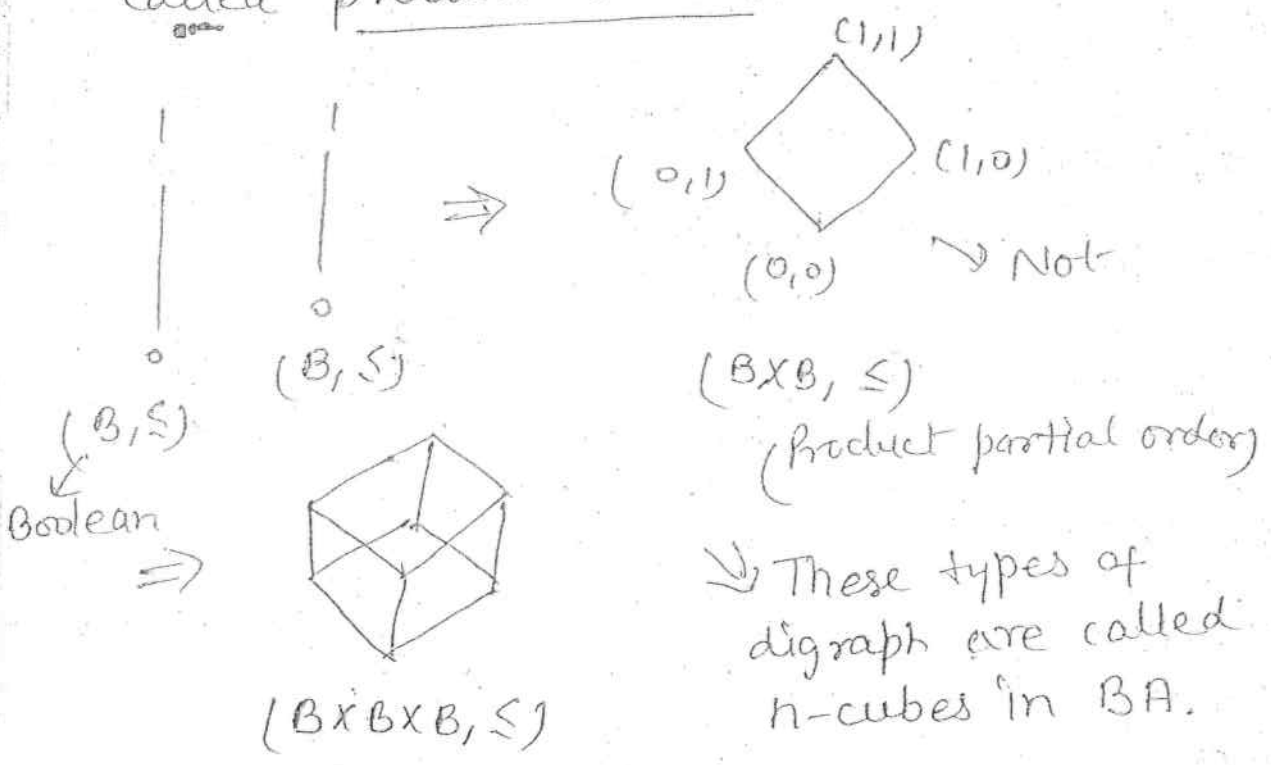
in boolean \Rightarrow idempotent & law of absorption also exists, but it is derived prop., so we don't include directly.

Types of lattice - (S, \leq)

- Bounded lattice - iff $0, 1 \in S$ for entire set.
- Complemented
- Distributive

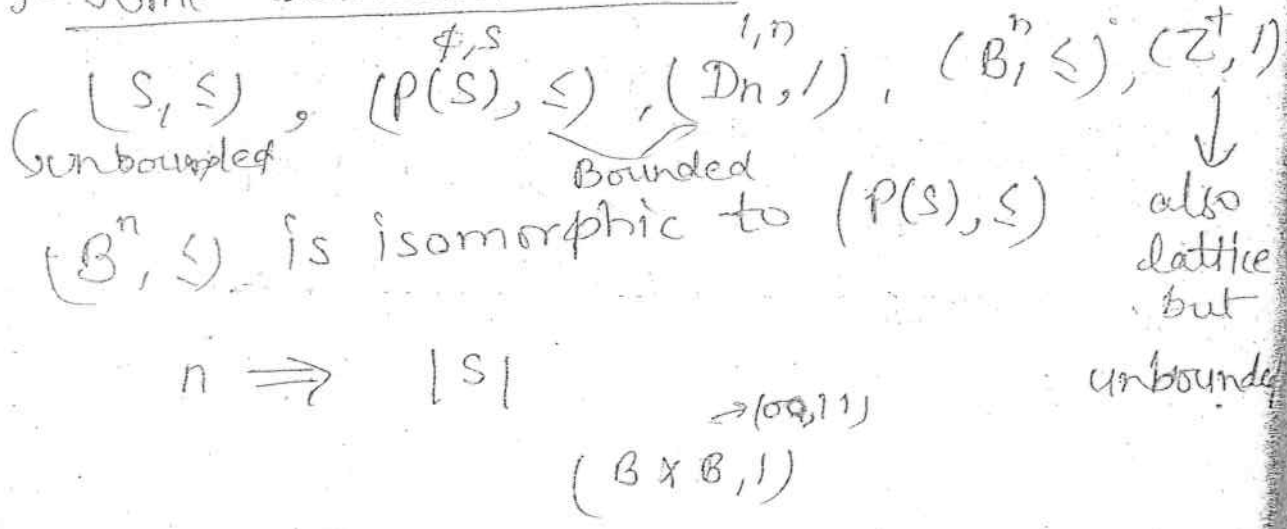
Lattice Results

1. Dual of any lattice always a lattice.
2. Product of two lattice also a lattice, called product lattice.



\Downarrow These types of digraph are called n-cubes in BA.

3. Some standard lattices



(S, \subseteq) is not bounded, so not BA.

set

$(D_n, |)$ is always lattice but not true for all for BA.

$(D_n, |)$ is BA iff $n = 2^3 3^2 \dots$

i.e. prime no. breakdown, all numbers should be unique.

so, $30 = 2 \times 3 \times 5 \rightarrow$ due to distinct, it is BA.

$20 = 2 \times 2 \times 5$
 \hookrightarrow since repetition, so not BA.

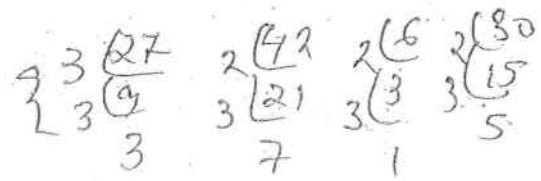
Which is BA \Rightarrow ?

$D_{30} = B.A.$

$D_6 = B.A.$

$D_{27} = \text{Not } B.A.$

$D_{42} = B.A.$



$(P(S), \subseteq) \rightarrow$ Not BA

$(D_n, |) \rightarrow$ Sometimes BA.

Toset is always a lattice, Toset also known as Chain.

order

ed

$(\mathbb{Z}^+, |)$

\downarrow
also lattice but bounded

$glb = ged \quad \& \quad lub = lcm$

Every finite lattices are Bounded.

If we have, $(a_1, a_2, a_3, \dots, a_n)$

$LUB = a_1 \vee a_2 \vee \dots \vee a_n$

$GLB = a_1 \wedge a_2 \wedge \dots \wedge a_n$

Prop. of bounded Lattice -

$$a \vee 0 = a$$

$$1 - \text{Identity prop.} \longrightarrow a \wedge 1 = a$$

$$2 - \text{Dominating prop.} \longrightarrow a \vee 1 = 1$$

$$a \wedge 0 = 0$$

$$3 - 0 \leq a \leq 1, \forall a \in S$$

* In bounded, minimum 2 elements are required always.

* In lattice, minimum 1 element is sufficient.

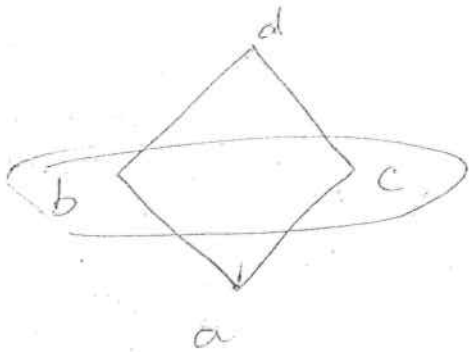
Complemented Lattice - Lattice L is complemented,

iff $\forall a \in L, a$ must have at least one complement.

$$\Rightarrow a \vee a' = 1$$

$$a \wedge a' = 0$$

$$0' = 1 \neq 1' = 0$$

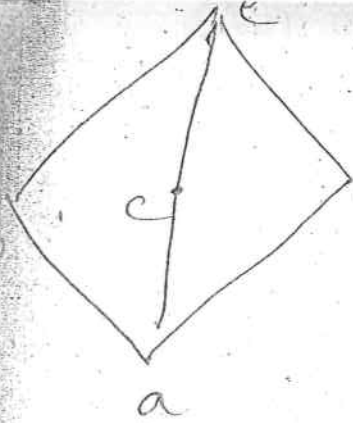


$$\Rightarrow$$

	comp.
a	d
b	c
c	b
d	a

as, complement never be related other than 0 & 1.

i.e. only incomparable can be complement.

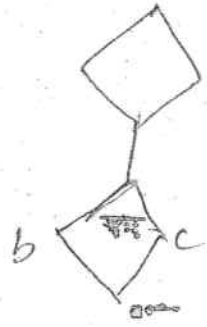


a	e
b	c, d
c	b, d
d	b, c
e	a

quired.

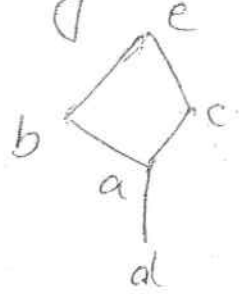
cient.

ented,



⇒ Does not have complement.

* In Neck type digram, there will not be any complemented.



⇒ as $bvc \Rightarrow a$ but we should get d. . .
also a have no complement.

ment



⇒ Not complement.

o x 1.

* A Tosel can be complemented, only if it has exactly two element.

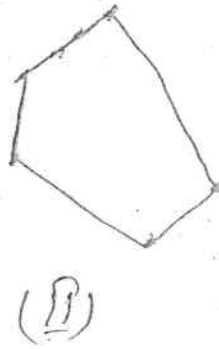
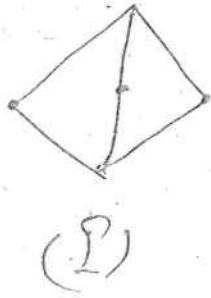
Distributive lattice - Lattice L is distri. iff

$\forall a, b, c \in S,$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

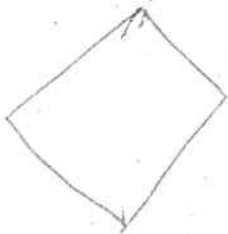
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

- A Lattice L is distrib: iff it does not contain sublattice isomorphic to one (P) or two (II) .



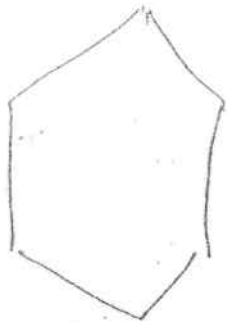
- These two are culprits, which destroys the prop. of distribution.

eg-



\Rightarrow ~~Not~~ distributive.

* Less than 5 pts. will always be distrib.



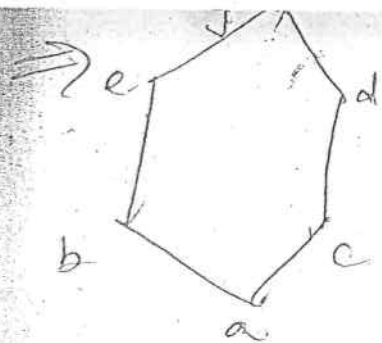
\Rightarrow Not distr. because pentagon is sublattice of this.

Sublattice - Same LUB & GLB must exist.

* Every ~~at~~ cycle with more than 5 pts. will not be distrib.

- If a complemented lattice is distrib. then complement will be unique always.

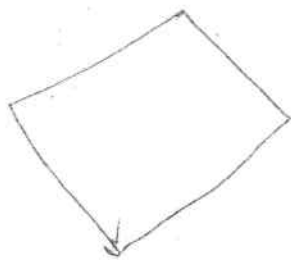
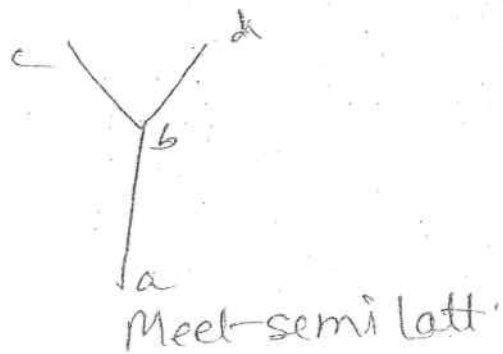
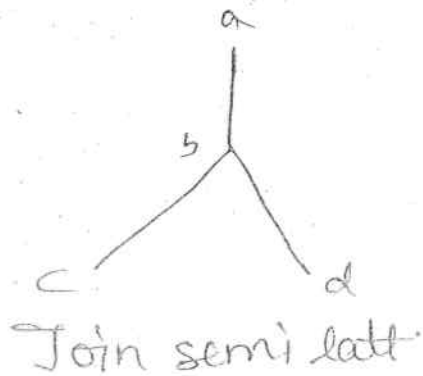
(P)



⇒ ∴ this complement but not have unique comp. hence it is not distri.

b's comp ⇒ {c, d}

Semi-lattice
 Meet-Semi lattice
 Join-Semi lattice



⇒ Join & Meet Semilattice both.

